Scaling limits for the exclusion process with a slow site

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Abstract

We consider the symmetric simple exclusion processes with a slow site in the discrete torus with n sites. In this model, particles perform nearest-neighbor symmetric random walks with jump rates everywhere equal to one, except at one particular site, *the slow site*, where the jump rate of entering that site is equal to one, but the jump rate of leaving that site is given by a parameter $g(n)$. Two cases are treated, namely $g(n) = 1 + o(1)$, and $g(n) = \alpha n^{-\beta}$ with $\beta > 1$, $\alpha > 0$. In the former, both the hydrodynamic behavior and equilibrium fluctuations are driven by the heat equation (with periodic boundary conditions when in finite volume). In the latter, they are driven by the heat equation with Neumann boundary conditions. We therefore establish the existence of a dynamical phase transition. The critical behavior remains open.
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1. Introduction

In the seventies, Dobrushin and Spitzer, see [22] and references therein, initiated the idea of obtaining a mathematically precise understanding of the emergence of macroscopic behavior

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in gases or fluids from the microscopic interaction of a large number of identical particles with stochastic dynamics. This approach has turned out to be extremely fruitful both in probability theory and statistical physics (e.g. see the books [24,13]) and it still raises attention nowadays. In this context, recent studies have been made in hydrodynamic limit/fluctuations of interacting particle systems in random/non homogeneous medium, see for instance [9,6,8,12] and references therein.

So far, most of the work done in this field concerns the bulk hydrodynamics, i.e., the derivation of macroscopic partial differential equations arising from the *bulk interactions* of the underlying particle system. To this end, one usually considers an infinite system or a finite torus with periodic boundary conditions and then takes the thermodynamic limit. However, in applications to physical systems one is usually confronted with finite systems, which requires the study of a partial differential equation on a finite interval with prescribed boundary conditions. This raises the question from which microscopic *boundary interactions* a given type of boundary condition emerges at the macroscopic scale.

This is an important issue both for boundary-driven open systems, where boundary interactions can induce long-range correlations [23] and bulk phase transitions due to the absence of particle conservation at the boundaries [3], and for bulk-driven conservative systems on the torus where even a single defect bond between two neighboring sites can change bulk relaxation behavior or lead to macroscopic discontinuities in the hydrostatic density profiles.[1] Given such rich behavior due to boundary effects in non-conservative or bulk-driven systems it is natural to explore the macroscopic role of a microscopic defect on a torus in a conservative system in the *absence* of bulk-driving and to ask whether such a defect can be described on macroscopic scale in terms of a boundary condition for the PDE describing the bulk hydrodynamics.

In this work we address this problem for the symmetric simple exclusion process (SSEP) on the discrete torus in the presence of a defect site. The model can be described as follows. Each site of the discrete torus with n sites, that we denote by $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$, is allowed to have at most one particle. To each site is associated a Poisson clock, all of them being independent. If there is a particle in the associated site, this particle chooses one of its nearest neighbors with equal probability when the clock rings. If the chosen site is empty, the particle jumps to it. Otherwise nothing happens. All sites have a Poisson clock of parameter two, except the origin, which has a Poisson clock of parameter $2g(n)$. If $g(n) < 1$, the origin behaves as a *trap*, and (in average) it keeps a particle there for a longer time than the other sites do. We call this site a *slow site*. The main results of the present work are the hydrodynamic limit and the equilibrium fluctuations for the exclusion process with such a slow site.

Specifically, for $g(n) = 1 + o(1)$ it is shown here that the limit for the time trajectory of the spatial density of particles is given by the solution of the heat equation with periodic boundary conditions, namely:

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in \mathbb{T}, \\ \rho(0, u) = \rho_0(u), & u \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is the one-dimensional continuous torus.

Moreover, considering the same particle system evolving on $\mathbb{Z}$, we prove that the equilibrium fluctuations of the system are driven by a generalized Ornstein–Uhlenbeck process $\mathcal{Y}_t$ which is

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[1] See [11,20,1,4,2,17] for numerical, exact and rigorous results for the asymmetric simple exclusion process and [21,19] for a review, including experimental applications of interacting particle systems with boundary interactions in physical and biological systems.

the solution of

$$dY_t = \Delta Y_t dt + \sqrt{2\chi(p)} \nabla dW_t$$

where W_t is a Brownian motion on the space $S'(\mathbb{R})$ of tempered distributions and $\chi(p)$ is the compressibility, which is a coefficient related to the invariant measure of the system. Both results are true irrespective of whether $o(1)$ is positive or negative, i.e., of whether the origin is a fast site or a slow site.

On the other hand, if $g(n) = \alpha n^{-\beta}$, $\alpha > 0$, $\beta > 1$, the limit for the time trajectory of the spatial density of particles is given by the solution of the heat equation with Neumann boundary conditions, namely:

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in (0, 1), \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = 0, & t \geq 0, \\ \rho(0, u) = \rho_0(u), & u \in (0, 1), \end{cases} \tag{1.2}$$

where 0^+ and 0^- denotes right and left side limits, respectively. This represents no passage of particles in the continuum limit.

Again considering the same particle system evolving on $\mathbb{Z}$, we prove that the equilibrium fluctuations of the system when $g(n) = \alpha n^{-\beta}$, $\alpha > 0$, $\beta > 1$ are driven by the solution of

$$dY_t = \Delta_{\text{Neu}} Y_t dt + \sqrt{2\chi(p)} \nabla_{\text{Neu}} dW_t,$$

which is essentially a version of the previous generalized Ornstein–Uhlenbeck process associated to the PDE (1.2), in the same setting of [8]. These Ornstein–Uhlenbeck processes are precisely stated in Section 2.

We point out that a similar model with conservative dynamics on the torus has been considered in [6,8,9], which consists in the SSEP with a *slow bond* of intensity $g(n) = \alpha n^{-\beta}$, $\alpha > 0$, $\beta \geq 0$. In that model particles perform nearest-neighbor symmetric random walks, whose jump rate is equal to one at all bonds, except at a particular *bond*, where it is equal to $g(n)$. From [6,8,9] it is known that hydrodynamic limit/fluctuations for exclusion processes with a slow bond have three different behaviors depending on the regime of β.

For the SSEP with a slow site that we treat here the methods used for the slow bond problem cannot be adapted in any straightforward fashion as the asymmetry at the slow site gives rise to novel difficulties in the study of its hydrodynamic behavior and fluctuations. The model is reversible, as is the SSEP with a slow bond, but it is not *self-dual, in contrast* to the SSEP with a slow bond. Moreover, the invariant measures for the SSEP with a slow site are not translation invariant, as happens for the SSEP with a slow bond. As a consequence, the proof of the hydrodynamic limit for the SSEP with a slow site requires different approaches from the ones of [6,8,9] and one cannot naively extend the results obtained for the slow bond to the case of the slow site.

As described above, in this paper we are able to characterize the hydrodynamic limit and the equilibrium fluctuations for $g(n) = \alpha n^{-\beta}$ when $\alpha > 0$, $\beta > 1$. The case $0 \leq \beta \leq 1$ remains open. However, we present and motivate a conjecture on the behavior of the system in that case. Moreover, since we present also the hydrodynamic limit and the equilibrium fluctuations for $g(n)$ close to one, namely $g(n) = 1 + o(1)$, the existence of a dynamical phase transition in the behavior of the system from periodic boundary conditions to Neumann boundary conditions at a critical value of β in the range $0 \leq \beta \leq 1$ is established.

In order to put our results into a broader perspective conservative particle systems with defects we point out that for a single slow bond the SSEP treated [6,8,9] exhibits the same hydrodynamic behavior as non-interacting random walks with a single slow bond. On the other hand, with a single slow site both hydrodynamic limit and fluctuations of non-interacting particles would be driven by a disconnect behavior for any $\beta > 0$, leading to Dirichlet boundary conditions with boundary densities 0. Thus the similarity between the SSEP and non-interacting particles that one finds for a slow bond breaks down for a slow site, adding further motivation for a detailed investigation of the SSEP with a slow site.

It is also worthwhile to compare our result with a result derived in [12] for a related problem. The model considered there is called the *Bouchaud* trap model. In that model, particles perform independent random walks in a random environment with traps given by i.i.d. alpha-stable random variables. In [12] it was proved that the hydrodynamic limit for such model is given by a generalized partial differential equation depending on an alpha-stable subordinator. The present paper suggests that a trap model of exclusion type should not have the same limit as obtained in [12] for a trap model of independent random walks. For the SSEP the asymmetry at the slow site yields a limit that has some properties in common with a slow bond and therefore a behavior completely different from the one observed in [12].

Here follows the outline of this paper. In Section 2 we give notations, precise definitions and statements of the results. In Section 3 we present the hydrodynamic limit of the model. In Section 4 we present the equilibrium fluctuations (in infinite volume). In Section 5 we state a conjecture on what should be the complete scenario for exclusion processes with a slow site. In Section 6 we present an extra result on the hydrodynamic behavior of the SSEP with k neighboring slow bonds, which we use as an argument to sustain our conjecture in Section 5.

2. Statement of results

2.1. The model

A particle system can be constructed through its generator or via Poisson processes. In this work we will make use of both.

Let $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n - 1\}$ be the one-dimensional discrete torus with n points. The simple symmetric exclusion process (SSEP) with a slow site is the Markov process with state space $\{0, 1\}^{\mathbb{T}_n}$ and with generator $\mathfrak{L}_n$ acting on functions $f : \{0, 1\}^{\mathbb{T}_n} \to \mathbb{R}$ as

$$\mathfrak{L}_n f(\eta) = \sum_{\substack{x,y \in \mathbb{T}_n \\ |x-y| \leq 1}} \xi_x^n \, \eta(x)(1 - \eta(y)) \left[f(\eta^{x,y}) - f(\eta) \right], \tag{2.1}$$

where the jump rates ξ_x^n are given by

$$\xi_x^n = \begin{cases} g(n), & \text{if } x = 0, \\ 1, & \text{if } x \in \mathbb{T}_n \setminus \{0\}, \end{cases}$$

where $g(n) > 0$ and $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x + 1)$. Formally,

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x + 1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x + 1, \\ \eta(y), & \text{otherwise.} \end{cases} \tag{2.2}$$
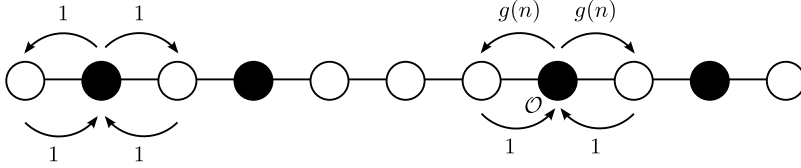
Fig. 1. Exclusion process with a *slow site*.

Its dynamics can be described as follows. To each site we attach two Poisson processes, one corresponding to jumps from x to $x + 1$ and the other corresponding to jumps from x to $x - 1$. If the site x is occupied and the site $x + 1$ is empty, the particle moves from site x to site $x + 1$ at a time arrival of the Poisson process associated to $\{x, x + 1\}$, and analogously for sites $\{x, x - 1\}$. The jump rates corresponding to those transitions are shown in Fig. 1.

For fixed n, let $\{\eta_\tau : \tau \geq 0\}$ be the Markov process with generator $\mathfrak{L}_n$. Notice that η_τ depends on $g(n)$, but we do not display this dependence in the notation. We denote by $\{\eta_t : t \geq 0\}$ the Markov process with generator $n^2 \mathfrak{L}_n$. This time factor n^2 is the so-called *diffusive time scaling*. We observe that this is equivalent to define $\eta_t := \eta_{n^2\tau}$.

Next we establish a family of invariant measures (in fact, reversible) for the dynamics introduced above.

Proposition 2.1. *For any* $p \in [0, 1]$*, the Bernoulli product measure* ν_p *on the space* $\{0, 1\}^{\mathbb{T}_n}$ *with marginals given by*

$$\nu_p\{\eta\,;\,\eta(x) = 1\} = m_p(x) = \begin{cases} \dfrac{\frac{p}{g(n)}}{(1 - p) + \frac{p}{g(n)}}, & \text{if } x = 0, \\[3mm] p, & \text{if } x \in \mathbb{T}_n \setminus \{0\}, \end{cases} \tag{2.3}$$

is reversible for the Markov process $\{\eta_\tau : \tau \geq 0\}$.

The proof of this proposition consists only in checking the detailed balance equation, which is straightforward and for that reason it will be omitted.

Above and in what follows, a sub-index in a function means a variable, *not a derivative*. Denote by $\mathbb{T}$ the one-dimensional continuous torus $\mathbb{R}/\mathbb{Z} = [0, 1)$ and by $\langle \cdot, \cdot \rangle$ the inner product in $L^2[0, 1]$.

2.2. Hydrodynamics

Definition 1. Let $\rho_0 : \mathbb{T} \to [0, 1]$ be a measurable function. We say that ρ is a weak solution of the heat equation with periodic boundary conditions given by

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in \mathbb{T}, \\ \rho(0, u) = \rho_0(u), & u \in \mathbb{T}, \end{cases} \tag{2.4}$$

if, for all $t \in [0, T]$ and for all $H \in C^2(\mathbb{T})$,

$$\langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_u^2 H_s \rangle ds = 0. \tag{2.5}$$

Next, we define what we mean by weak solutions of the heat equation with Neumann boundary conditions, as given in (1.2). We introduce first some technical background on Sobolev spaces.

Definition 2. Let $\mathcal{H}^1$ be the set of all L^1 functions $\zeta : [0, 1] \to \mathbb{R}$ such that there exists a function $\partial_u \zeta \in L^2[0, 1]$ satisfying

$$\langle \partial_u G, \zeta \rangle = -\langle G, \partial_u \zeta \rangle,$$

for all $G \in C^\infty[0, 1]$ with compact support contained in $(0, 1)$. For $\zeta \in \mathcal{H}^1$, we define the norm

$$\|\zeta\|_{\mathcal{H}^1} := \left(\|\zeta\|_{L^2[0,1]}^2 + \|\partial_u \zeta\|_{L^2[0,1]}^2 \right)^{1/2}.$$

Let $L^2(0, T; \mathcal{H}^1)$ be the space of all measurable functions $\xi : [0, T] \to \mathcal{H}^1$ such that

$$\|\xi\|_{L^2(0,T;\mathcal{H}^1)}^2 := \int_0^T \|\xi_t\|_{\mathcal{H}^1}^2 \, dt < \infty.$$

Abusing notation slightly, we denote by $C^2[0, 1]$ the set of functions $H : \mathbb{T} \to \mathbb{R}$ that are continuously twice differentiable in $\mathbb{T} \setminus \{0\}$ and have a C^2-extension to the closed interval $[0, 1]$.

Definition 3. Let $\rho_0 : \mathbb{T} \to [0, 1]$ be a measurable function. We say that ρ is a weak solution of the heat equation with Neumann boundary conditions

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in (0, 1), \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = 0, & t \geq 0, \\ \rho(0, u) = \rho_0(u), & u \in (0, 1), \end{cases} \tag{2.6}$$

if ρ belongs to $L^2(0, T; \mathcal{H}^1)$ and for all $t \in [0, T]$ and for all $H \in C^2[0, 1]$,

$$\langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t \left(\rho_s(0^+) \, \partial_u H(0^+) - \rho_s(0^-) \, \partial_u H(0^-) \right) ds = 0.$$

Let $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{T}_n})$ be the path space of càdlàg[2] trajectories with values in $\{0, 1\}^{\mathbb{T}_n}$. For a measure μ_n on $\{0, 1\}^{\mathbb{T}_n}$, denote by $\mathbb{P}_{\mu_n}$ the probability measure on $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{T}_n})$ induced by the initial state μ_n and the Markov process $\{\eta_t : t \geq 0\}$. Notice that in fact $\mathbb{P}_{\mu_n} = \mathbb{P}_{\mu_n}^{g(n),n}$ but we will not carry the dependence on n nor g in order to not overload notation. By $\mathbb{E}_{\mu_n}$ we mean the expectation with respect to $\mathbb{P}_{\mu_n}$.

The notation $\eta_\cdot$ is reserved to represent elements of the Skorohod space $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{T}_n})$, i.e., time trajectories of the exclusion process with a slow site. This notation $\eta_\cdot$ should not be confused with the notation η for elements of $\{0, 1\}^{\mathbb{T}_n}$.

From now on we fix a profile $\gamma : \mathbb{T} \to [0, 1]$, representing the initial density of particles. To avoid uninteresting technical complications, we assume that γ is continuous at all $x \in \mathbb{T} \setminus \{0\}$ and bounded from below by a positive constant:

$$\zeta := \inf_{x \in \mathbb{T}} \gamma(x) > 0. \tag{2.7}$$

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[2] From the French, "continuous from the right with limits from the left".

Theorem 2.2. *For each $n \in \mathbb{N}$, let μ_n be a Bernoulli product measure on $\{0, 1\}^{\mathbb{T}_n}$ with marginal distributions given by*

$$\mu_n\{\eta ; \ \eta(x) = 1\} = \gamma\left(\frac{x}{n}\right). \tag{2.8}$$

Then, for any $t > 0$, for every $\delta > 0$ and every $H \in C(\mathbb{T})$, it holds that

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\left\{\eta. : \left|\frac{1}{n}\sum_{x\in\mathbb{T}_n} H\left(\frac{x}{n}\right)\eta_t(x) - \int_{\mathbb{T}} H(u)\,\rho(t,u)du\right| > \delta\right\} = 0, \tag{2.9}$$

where

- *for $g(n) = 1 + o(1)$, ρ is the unique weak solution of (2.4);*
- *for $g(n) = \alpha n^{-\beta}$, $\alpha > 0$, $\beta > 1$, ρ is the unique weak solution of (2.6);*

and where, in both cases, the initial condition of the corresponding partial differential equation is given by $\rho_0 = \gamma$.

Remark 2.3. About the constants: To avoid repetitions along the paper, we fix, once and for all, the assumptions $\alpha > 0$ and $\beta > 1$.

Remark 2.4. About the statement of the Theorem: If at the initial time the density of particles converges to the profile $\gamma(\cdot)$, then, in the future time t, the density of particles converges to a profile $\rho(t, \cdot)$ which is the weak solution of the heat equation with the corresponding boundary conditions and with initial condition $\rho_0 = \gamma$.

Remark 2.5. About the scaling: In the claim of Theorem 2.2 one can see that the space is rescaled by n^{-1} (space between sites) and time is rescaled by n^2, since the "future time" is indeed tn^2. This is the *diffusive* time scaling.

Remark 2.6. About the initial measure: We can weaken the hypothesis on μ_n by dropping the condition of being a product measure, and assuming that $\{\mu_n\}_{n\in\mathbb{N}}$ is *associated to* $\gamma(\cdot)$, see [13]. In that case the statement of Theorem 2.2 remains in force. However, this hypothesis would complicate the attractiveness tools at Section 3.5 and for this reason we assume (2.8).

Remark 2.7. About the weak solution: The weak solution of (2.6) is a function defined on the interval $[0, 1]$, not on the torus. But, as already explained, Lebesgue almost sure, it is the same. Thus it makes sense to integrate ρ in the torus $\mathbb{T}$, as it appears in Eq. (2.9).

2.3. Equilibrium density fluctuations

In this section we consider η_t evolving on the one-dimensional lattice $\mathbb{Z}$ and starting from the invariant state ν_p, with $p \in (0, 1)$. Therefore, the generator of the process is given by (2.1) with $\mathbb{T}_n$ replaced by $\mathbb{Z}$, namely

$$\mathcal{L}_n f(\eta) = \sum_{\substack{x,y\in\mathbb{Z} \\ |x-y|\leq 1}} \xi_x^n \eta(x)(1 - \eta(y))\left[f(\eta^{x,y}) - f(\eta)\right],$$

for local functions $f : \{0, 1\}^{\mathbb{Z}} \to \mathbb{R}$.

From now on we fix $p \in (0, 1)$. In order to establish the central limit theorem (C.L.T.) for the density under the invariant state ν_p, we need to introduce the fluctuation field as the linear functional acting on test functions H as

$$\mathcal{Y}_t^n(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\left(\frac{x}{n}\right)(\eta_t(x) - m_p(x)), \tag{2.10}$$

where $m_p(x)$ is the mean of $\eta_t(x)$ with respect to ν_p introduced in (2.3). We emphasize that $\{\eta_t : t \geq 0\}$ is the Markov process with generator $n^2 \mathfrak{L}_n$. Denote by $\mathbb{P}_p$ the probability measure on the Skorohod path space $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{Z}})$ induced by the initial state ν_p and the Markov process $\{\eta_t : t \geq 0\}$ and we denote by $\mathbb{E}_p$ the expectation with respect to $\mathbb{P}_p$.

Now we introduce the space of test functions. Since the hydrodynamics is governed by different partial differential equations, the state space for H depends on the jump rate $g(n)$ that we defined at the slow site.

Definition 4. Let $\mathcal{S}(\mathbb{R})$ be the usual Schwartz space of functions $H : \mathbb{R} \to \mathbb{R}$ such that $H \in C^\infty(\mathbb{R})$ and

$$\|H\|_{k,\ell} := \sup_{x \in \mathbb{R}} \left| (1 + |x|^\ell) \frac{d^k H}{dx^k}(x) \right| < \infty,$$

for all integers $k, \ell \geq 0$. We define $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ as the space composed of functions $H : \mathbb{R} \to \mathbb{R}$ such that

(1) Except possibly at $x = 0$, the function H is continuous and infinitely differentiable,
(2) The function H is continuous from the right at zero,
(3) For all integers $k, \ell \geq 0$,

$$\|H\|_{k,\ell,+} := \sup_{x > 0} \left| (1 + |x|^\ell) \frac{d^k}{dx^k} H(x) \right| < \infty,$$

and

$$\|H\|_{k,\ell,-} := \sup_{x < 0} \left| (1 + |x|^\ell) \frac{d^k H}{dx^k}(x) \right| < \infty.$$

(4) For any integer $k \geq 0$,

$$\lim_{x \to 0^+} \frac{d^{2k+1} H}{dx^{2k+1}}(x) = \lim_{x \to 0^-} \frac{d^{2k+1} H}{dx^{2k+1}}(x) = 0.$$

Notice that it is not required that H is continuous at $x = 0$. Intuitively, this space $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ corresponds to two independent Schwartz spaces in each half line. The chosen notation comes from the expression *Neumann boundary conditions*.

Both spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ are Fréchet spaces. The proof that $\mathcal{S}(\mathbb{R})$ is Fréchet can be found in [18], for instance. The proof that $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ is Fréchet is quite similar and will be omitted.

The set of continuous linear functions $f : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ and $f : \mathcal{S}_{\mathrm{Neu}}(\mathbb{R}) \to \mathbb{R}$ with respect to the topology generated by the corresponding semi-norms will be denoted by $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}'_{\mathrm{Neu}}(\mathbb{R})$, respectively.

The notation ∇ and Δ mean the first and second space derivatives. In the case of $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$, we will make use of the following definition:

Definition 5. We define the operators $\nabla_{\text{Neu}} : \mathcal{S}_{\text{Neu}}(\mathbb{R}) \to \mathcal{S}_{\text{Neu}}(\mathbb{R})$ and $\Delta_{\text{Neu}} : \mathcal{S}_{\text{Neu}}(\mathbb{R}) \to \mathcal{S}_{\text{Neu}}(\mathbb{R})$ by

$$\nabla_{\text{Neu}} H(u) = \begin{cases} \dfrac{dH}{du}(u), & \text{if } u \neq 0, \\ \lim_{u \to 0^+} \dfrac{dH}{du}(u), & \text{if } u = 0, \end{cases}$$

$$\Delta_{\text{Neu}} H(u) = \begin{cases} \dfrac{d^2 H}{du^2}(u), & \text{if } u \neq 0, \\ \lim_{u \to 0^+} \dfrac{d^2 H}{du^2}(u), & \text{if } u = 0. \end{cases}$$

Notice that these operators are essentially the first and second space derivatives, but defined in specific domains, which changes the meaning of the operator. Roughly speaking, the operator Δ_{Neu} is the operator associated to a system blocked at the origin.

2.4. Ornstein–Uhlenbeck process

Denote by $\chi(p) = p(1 - p)$ the so-called *static compressibility* of the system. Based on [10,13], we have a characterization of the generalized Ornstein–Uhlenbeck process, which is a solution of

$$d\mathcal{Y}_t = \Delta \mathcal{Y}_t dt + \sqrt{2\chi(p)} \nabla d\mathcal{W}_t, \tag{2.11}$$

where $d\mathcal{W}_t$ is a space–time white noise of unit variance, in terms of a martingale problem. We will see later that this process, which take values on $\mathcal{S}'(\mathbb{R})$, governs the equilibrium fluctuations of the density of particles when the strength of the slow site is given by $g(n) = 1 + o(1)$.

On the other hand, when the strength is given by $g(n) = \alpha n^{-\beta}$, the corresponding Ornstein–Uhlenbeck process will be the solution of

$$d\mathcal{Y}_t = \Delta_{\text{Neu}} \mathcal{Y}_t dt + \sqrt{2\chi(p)} \nabla_{\text{Neu}} d\mathcal{W}_t, \tag{2.12}$$

and taking values on $\mathcal{S}'_{\text{Neu}}(\mathbb{R})$.

In what follows $\mathcal{D}([0, T], \mathcal{S}'(\mathbb{R}))$ (resp. $\mathcal{C}([0, T], \mathcal{S}'(\mathbb{R}))$) is the space of càdlàg (resp. continuous) $\mathcal{S}'(\mathbb{R})$ valued functions endowed with the Skorohod topology. Analogous definitions hold for $\mathcal{D}([0, T], \mathcal{S}'_{\text{Neu}}(\mathbb{R}))$ and $\mathcal{C}([0, T], \mathcal{S}'_{\text{Neu}}(\mathbb{R}))$.

The rigorous meaning of Eqs. (2.11) and (2.12) is given in terms of the two next propositions. Denote by $T_t : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ the semi-group of the heat equation in the line (see [7] for instance). It is well known that

Proposition 2.8. *There exists an unique random element $\mathcal{Y}$. taking values in the space $\mathcal{C}([0, T], \mathcal{S}'(\mathbb{R}))$ such that:*

(i) *For every function $H \in \mathcal{S}(\mathbb{R})$, $\mathcal{M}_t(H)$ and $\mathcal{N}_t(H)$, given by*

$$\mathcal{M}_t(H) = \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\Delta H) ds,$$
$$\mathcal{N}_t(H) = \left(\mathcal{M}_t(H)\right)^2 - 2\chi(p)\, t\, \|\nabla H\|_{L^2(\mathbb{R})}^2, \tag{2.13}$$

are $\mathcal{F}_t$-martingales, where for each $t \in [0, T]$, $\mathcal{F}_t := \sigma(\mathcal{Y}_s(H); s \leq t, H \in \mathcal{S}(\mathbb{R}))$.

(ii) $\mathcal{Y}_0$ *is a Gaussian field of mean zero and covariance given on* $G, H \in \mathcal{S}(\mathbb{R})$ *by*

$$\mathbb{E}_p\big[\mathcal{Y}_0(G)\mathcal{Y}_0(H)\big] = \chi(p) \int_{\mathbb{R}} G(u)H(u)du. \tag{2.14}$$

Moreover, for each $H \in \mathcal{S}(\mathbb{R})$, *the stochastic process* $\{\mathcal{Y}_t(H) : t \geq 0\}$ *is Gaussian, being the distribution of* $\mathcal{Y}_t(H)$ *conditionally to* $\mathcal{F}_s$, *for* $s < t$, *normal of mean* $\mathcal{Y}_s(T_{t-s}H)$ *and variance* $\int_0^{t-s} \|\nabla T_r H\|^2_{L^2(\mathbb{R})} dr.$

We call the random element $\mathcal{Y}.$ the generalized Ornstein–Uhlenbeck process of characteristics ∇ and Δ. From the second equation in (2.13) and Lévy's Theorem on the martingale characterization of Brownian motion, the process

$$\big(2\chi(p)\|\nabla H\|^2_{L^2(\mathbb{R})}\big)^{-1/2} \mathcal{M}_t(H) \tag{2.15}$$

is a standard Brownian motion. Therefore, in view of Proposition 2.8, it makes sense to say that $\mathcal{Y}.$ is the formal solution of (2.11).

Now, let $T_t^{\text{Neu}} : \mathcal{S}_{\text{Neu}}(\mathbb{R}) \to \mathcal{S}_{\text{Neu}}(\mathbb{R})$ be the semi-group associated to the following partial differential equation with Neumann boundary conditions:

$$\begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x), & t \geq 0, \ x \in \mathbb{R} \setminus \{0\} \\ \partial_x u(t, 0^+) = \partial_x u(t, 0^-) = 0 & t \geq 0 \\ u(0, x) = H(x), & x \in \mathbb{R}. \end{cases} \tag{2.16}$$

See [7] for an explicit expression of T_t^{Neu}. In a similar way, we have

Proposition 2.9 (*See [7]*)**.** *There exists an unique random element* $\mathcal{Y}.$ *taking values in the space* $\mathcal{C}([0, T], \mathcal{S}'_{\text{Neu}}(\mathbb{R}))$ *such that:*

(i) *For every function* $H \in \mathcal{S}_{\text{Neu}}(\mathbb{R})$, $\mathcal{M}_t(H)$ *and* $\mathcal{N}_t(H)$ *given by*

$$\mathcal{M}_t(H) = \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\Delta_{\text{Neu}}H)ds,$$
$$\mathcal{N}_t(H) = \big(\mathcal{M}_t(H)\big)^2 - 2\chi(p)\, t\, \|\nabla_{\text{Neu}}H\|^2_{L^2(\mathbb{R})} \tag{2.17}$$

are $\mathcal{F}_t$-*martingales, where for each* $t \in [0, T]$, $\mathcal{F}_t := \sigma(\mathcal{Y}_s(H); s \leq t, H \in \mathcal{S}_{\text{Neu}}(\mathbb{R}))$.

(ii) $\mathcal{Y}_0$ *is a Gaussian field of mean zero and covariance given on* $G, H \in \mathcal{S}_{\text{Neu}}(\mathbb{R})$ *by* (2.14).

Moreover, for each $H \in \mathcal{S}_{\text{Neu}}(\mathbb{R})$, *the stochastic process* $\{\mathcal{Y}_t(H); t \geq 0\}$ *is Gaussian, being the distribution of* $\mathcal{Y}_t(H)$ *conditionally to* $\mathcal{F}_s$, *for* $s < t$, *normal of mean* $\mathcal{Y}_s(T_{t-s}^{\text{Neu}}H)$ *and variance* $\int_0^{t-s} \|\nabla T_r^{\text{Neu}}H\|^2_{L^2(\mathbb{R})} dr.$

We call the random element $\mathcal{Y}.$ the generalized Ornstein–Uhlenbeck process of characteristics ∇_{Neu} and Δ_{Neu}. We are in position to state our result for the fluctuations of the density of particles.

Theorem 2.10 (*C.L.T. for the Density of Particles*)**.** *Consider the Markov process* $\{\eta_t : t \geq 0\}$ *starting from the invariant state* v_p *under the assumption* $g(n) = 1 + o(1)$. *Then, the sequence of processes* $\{\mathcal{Y}_t^n\}_{n \in \mathbb{N}}$ *converges in distribution, as* $n \to +\infty$, *with respect to the Skorohod topology of* $\mathcal{D}([0, T], \mathcal{S}'(\mathbb{R}))$ *to* $\mathcal{Y}_t$ *in* $\mathcal{C}([0, T], \mathcal{S}'(\mathbb{R}))$, *the generalized Ornstein–Uhlenbeck process of characteristics* Δ, ∇ *which is the formal solution of the Eq.* (2.11).

On the other hand, if we consider $g(n) = \alpha n^{-\beta}$, $\alpha > 0$ and $\beta > 1$, then $\{\mathcal{Y}_t^n\}_{n \in \mathbb{N}}$ converges in distribution, as $n \to +\infty$, with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathcal{S}'_{Neu}(\mathbb{R}))$ to $\mathcal{Y}_t$ in $\mathcal{C}([0, T], \mathcal{S}'_{Neu}(\mathbb{R}))$, the generalized Ornstein–Uhlenbeck process of characteristics Δ_{Neu}, ∇_{Neu} which is the formal solution of Eq. (2.12).

3. Hydrodynamics

We proceed to define the spatial density of particles of the exclusion process, where we embed the discrete torus $\mathbb{T}_n$ in the continuous torus $\mathbb{T}$.

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one, endowed with the weak topology. Let $\pi_t^n \in \mathcal{M}$ be the measure on $\mathbb{T}$ obtained by rescaling time by n^2, rescaling space by n^{-1}, and assigning mass n^{-1} to each particle, i.e.,

$$\pi_t^n(\eta, du) = \frac{1}{n} \sum_{x \in \mathbb{T}_n} \eta_t(x)\, \delta_{x/n}(du), \tag{3.1}$$

where δ_u is the Dirac measure concentrated on u. The usual name for $\pi_t^n(\eta, du)$ is *empirical measure*. For an integrable function $H : \mathbb{T} \to \mathbb{R}$, the expression $\langle \pi_t^n, H \rangle$ stands for the integral of H with respect to π_t^n:

$$\langle \pi_t^n, H \rangle = \frac{1}{n} \sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \eta_t(x).$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{T})$. Also, when π_t has a density ρ, namely when $\pi(t, du) = \rho(t, u)du$, we sometimes write $\langle \rho_t, H \rangle$ for $\langle \pi_t, H \rangle$.

To avoid unwanted topological issues, in the entire paper a time horizon $T > 0$ is fixed. Let $\mathcal{D}([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0, T] \to \mathcal{M}$ endowed with the *Skorohod* topology. For each probability measure μ_n on $\{0, 1\}^{\mathbb{T}_n}$, denote by $\mathbb{Q}_{\mu_n}^n$ the measure on the path space $\mathcal{D}([0, T], \mathcal{M})$ induced by the measure μ_n and the process π_t^n introduced in (3.1).

Recall the profile $\gamma : \mathbb{T} \to [0, 1]$ and the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of measures on $\{0, 1\}^{\mathbb{T}_n}$ defined through (2.8). Let $\mathbb{Q}$ be the probability measure on the space $\mathcal{D}([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, du) = \rho(t, u)du$, where

- if $g(n) = 1 + o(1)$, the function ρ is the unique weak solution of (2.4);
- if $g(n) = \frac{\alpha}{n^\beta}$, the function ρ is the unique weak solution of (2.6).

Proposition 3.1. *Considering the two possibilities above for the function g, the sequence of probability measures $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ converges weakly to $\mathbb{Q}$ as $n \to \infty$.*

Since Theorem 2.2 is an immediate corollary of the previous proposition, our goal is to prove Proposition 3.1.

The proof is divided in several parts. In Section 3.1, we show that the sequence $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ is tight. In Sections 3.4 and 3.6, we show that, for each case of g, $\mathbb{Q}$ is the only possible limit along subsequences of $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$. This assures that the sequence $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ converges weakly to $\mathbb{Q}$, as $n \to \infty$.

3.1. Tightness

In order to prove tightness of $\{\pi_t^n : 0 \le t \le T\}_{n \in \mathbb{N}}$ it is enough to show tightness of the real-valued processes $\{\langle \pi_t^n, H \rangle : 0 \le t \le T\}_{n \in \mathbb{N}}$ for a set of functions $H \in C(\mathbb{T})$, provided

this set of functions is dense in $C(\mathbb{T})$ with respect to the uniform topology (see [13, page 54, Proposition 1.7]). For that purpose, let $H \in C^2(\mathbb{T})$. By Dynkin's formula,

$$M_t^n(H) := \langle \pi_t^n, H \rangle - \langle \pi_0^n, H \rangle - \int_0^t n^2 \mathfrak{L}_n \langle \pi_s^n, H \rangle \, ds \qquad (3.2)$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \le t)$. Moreover,

$$\left(M_t^n(H) \right)^2 - \int_0^t \left(n^2 \mathfrak{L}_n [\langle \pi_s^n, H \rangle]^2 - 2 \langle \pi_s^n, H \rangle \, n^2 \mathfrak{L}_n \langle \pi_s^n, H \rangle \right) ds \qquad (3.3)$$

is also a martingale with respect to the same filtration, see [13]. In order to prove tightness of $\{\pi_t^n(H) : 0 \le t \le T\}_{n \in \mathbb{N}}$, we shall prove tightness of each term in the formula above and then we invoke the fact that a sequence of a finite sum of tight processes is again tight.

Since (3.3) is a martingale, doing elementary calculations we obtain the quadratic variation of $M_t^n(H)$ at time T as

$$\langle M^n(H) \rangle_T = \int_0^T \sum_{x \in \mathbb{T}_n \setminus \{0\}} \left((\eta_s(x) - \eta_s(x+1)) \left(H \left(\frac{x+1}{n} \right) - H \left(\frac{x}{n} \right) \right) \right)^2 ds$$

$$+ \int_0^T \left(\eta_s(1)(1 - \eta_s(0)) + g(n) \eta_s(0)(1 - \eta_s(1)) \right)$$

$$\times \left(H \left(\frac{1}{n} \right) - H \left(\frac{0}{n} \right) \right)^2 ds$$

$$+ \int_0^T \left(\eta_s(-1)(1 - \eta_s(0)) + g(n) \eta_s(0)(1 - \eta_s(-1)) \right)$$

$$\times \left(H \left(\frac{-1}{n} \right) - H \left(\frac{0}{n} \right) \right)^2 ds. \qquad (3.4)$$

The smoothness of H implies that $\lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[\langle M^n(H) \rangle_T \right] = 0$. Hence $M_T^n(H)$ converges to zero in $L^2(\mathbb{P}_{\mu_n})$ as $n \to \infty$ and, by Doob's inequality, for every $\delta > 0$,

$$\lim_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \eta. : \sup_{0 \le t \le T} |M_t^n(H)| > \delta \right\} = 0. \qquad (3.5)$$

In particular, this yields tightness of the sequence of martingales $\{M_t^n(H) : 0 \le t \le T\}_{n \in \mathbb{N}}$.

A long computation, albeit completely elementary, shows us that the term $n^2 \mathfrak{L}_n \langle \pi_s^n, H \rangle$ appearing inside the time integral in (3.2) can be rewritten as

$$n \sum_{x \in \mathbb{T}_n \setminus \{0\}} \left(H \left(\frac{x+1}{n} \right) + H \left(\frac{x-1}{n} \right) - 2H \left(\frac{x}{n} \right) \right) \eta_s(x)$$

$$+ n g(n) \left(H \left(\frac{1}{n} \right) + H \left(\frac{-1}{n} \right) - 2H \left(\frac{0}{n} \right) \right) \eta_s(0)$$

$$+ n(1 - g(n)) \left[\left(H \left(\frac{-1}{n} \right) - H \left(\frac{0}{n} \right) \right) \eta_s(0) \, \eta_s(-1) \right.$$

$$+ \left. \left(H \left(\frac{1}{n} \right) - H \left(\frac{0}{n} \right) \right) \eta_s(0) \, \eta_s(1) \right]. \qquad (3.6)$$

We note that the first term above corresponds to the discrete Laplacian leading to the heat equation, while the other two terms arise from the boundary conditions.

By the smoothness of H again, there exists a constant $c_H > 0$ such that $|n^2 \mathfrak{L}_n \langle \pi_s^n, H \rangle| \leq c_H$, which in turn gives

$$\left| \int_r^t n^2 \mathfrak{L}_n \langle \pi_s^n, H \rangle ds \right| \leq c_H \, |t - r|.$$

By the Arzelà–Ascoli Theorem the sequence of these integral terms is a relatively compact set, with respect to the uniform topology, therefore it is tight. The term $\langle \pi_0^n, H \rangle$ is constant in time and bounded, thus is tight as well. This concludes the proof that the set of measures $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ is tight.

3.2. Entropy

Denote by $\mathbf{H}(\mu | \nu_p)$ the entropy of a probability measure μ with respect to the invariant state ν_p. For a precise definition and properties of the entropy, we refer the reader to [13].

Proposition 3.2. *There exists a finite constant* $K_0 := K_0(p)$, *such that*

$$\mathbf{H}(\mu | \nu_p) \leq K_0 \, n,$$

for any probability measure μ *on* $\{0, 1\}^{\mathbb{T}_n}$.

Proof. Recall the definition of ν_p and notice that

$$\mathbf{H}(\mu | \nu_p) = \sum_{\eta \in \{0,1\}^{\mathbb{T}_n}} \mu(\eta) \log \Big(\frac{\mu(\eta)}{\nu_p(\eta)} \Big) \leq \sum_{\eta \in \{0,1\}^{\mathbb{T}_n}} \mu(\eta) \log \Big(\frac{1}{\nu_p(\eta)} \Big).$$

Recall (2.3). By the assumption $0 < p < 1$ and the inequality

$$\nu_p(\eta) \geq (p \wedge (1 - p))^{n-1} (m_p(0) \wedge (1 - m_p(0)))$$

we conclude that $\mathbf{H}(\mu | \nu_p) \leq K_0 \, n$ for some $K_0 > 0$ depending only on p. $\quad \square$

A remark: In particular, the estimate $\mathbf{H}(\mu_n | \nu_p) \leq K_0 \, n$ holds for the measures μ_n defined in (2.8).

3.3. Dirichlet form

Let f be any density with respect to the invariant measure ν_p. In others words, f is a non negative function $f : \{0, 1\}^{\mathbb{T}_n} \to \mathbb{R}$ satisfying $\int f(\eta) \nu_p(d\eta) = 1$. The Dirichlet form $\mathfrak{D}_n$ is the convex and lower semicontinuous functional defined through

$$\mathfrak{D}_n(\sqrt{f}) = - \int \sqrt{f(\eta)} \, \mathfrak{L}_n \sqrt{f(\eta)} \, \nu_p(d\eta).$$

Invoking a general result [13, Appendix 1, Prop. 10.1] we can write $\mathfrak{D}_n$ as

$$\mathfrak{D}_n(\sqrt{f}) = \frac{g(n)}{2} \int \eta(0)(1 - \eta(1)) \Big\{ \sqrt{f(\eta^{0,1})} - \sqrt{f(\eta)} \Big\}^2 \nu_p(d\eta)$$

$$+ \frac{1}{2} \int \eta(1)(1 - \eta(0)) \Big\{ \sqrt{f(\eta^{0,1})} - \sqrt{f(\eta)} \Big\}^2 \nu_p(d\eta)$$

$$+ \frac{1}{2} \int \eta(-1)(1 - \eta(0)) \left\{ \sqrt{f(\eta^{0,-1})} - \sqrt{f(\eta)} \right\}^2 \nu_p(d\eta)$$

$$+ \frac{g(n)}{2} \int \eta(0)(1 - \eta(-1)) \left\{ \sqrt{f(\eta^{0,-1})} - \sqrt{f(\eta)} \right\}^2 \nu_p(d\eta)$$

$$+ \frac{1}{2} \sum_{\substack{x \in \mathbb{T}_n \\ x \neq 0, -1}} \int \left\{ \sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right\}^2 \nu_p(d\eta), \tag{3.7}$$

where $\eta^{x,x+1}$ has been defined in (2.2).

Proposition 3.3. *Let $\mathbb{Q}^*$ be a limit of a subsequence of the sequence of probabilities measures $\{\mathbb{Q}^n_{\mu_n}\}_{n \in \mathbb{N}}$. Then $\mathbb{Q}^*$ is concentrated on trajectories $\pi(t, du)$ with a density with respect to the Lebesgue measure, i.e., of the form $\pi(t, du) = \rho(t, u)du$. Moreover, the density $\rho(t, u)$ belongs to the space $L^2(0, T; \mathcal{H}^1)$, see Definition 2.*

The proof of the proposition above can be adapted from [6, Proposition 5.6] and for this reason it will be omitted. For the interested reader, we briefly indicate some steps of this adaptation.

We begin by observing that [6, Proposition 5.6] is in fact a consequence of [6, Lemma 5.8]. Thus we just describe how to prove, in our case, the statement in [6, Lemma 5.8].

There are two basic ingredients in the proof of [6, Lemma 5.8]. The first one is that the entropy (with respect to the invariant measure) of any probability measure on the state space of the process, namely, $\{0, 1\}^{\mathbb{T}_n}$ does not grow more than linearly. In our case, this result is proved in Proposition 3.2.

The second ingredient in the proof of [6, Lemma 5.8] is the fact that, except at the defect, the Dirichlet form of the considered process coincides with the Dirichlet form of the homogeneous exclusion process. This fact indeed holds for the exclusion process with a slow site and therefore the same proof of [6, Lemma 5.8] applies here.

3.4. Hydrodynamic limit for $g(n) = 1 + o(1)$

Proof of Proposition 3.1 for $g(n) = 1 + o(1)$. Let $\mathbb{Q}^*$ be the weak limit of some convergent subsequence $\{\mathbb{Q}^{n_j}_{\mu_{n_j}}\}_{j \in \mathbb{N}}$ of the sequence $\{\mathbb{Q}^n_{\mu_n}\}_{n \in \mathbb{N}}$. In order not to overburden the notation, denote this subsequence just by $\{\mathbb{Q}^n_{\mu_n}\}_{n \in \mathbb{N}}$. By Proposition 3.3, the probability measure $\mathbb{Q}^*$ is concentrated on trajectories $\pi(t, du) = \rho(t, u)du$ such that $\rho(t, u) \in L^2(0, T; \mathcal{H}^1)$. Our goal is to conclude that ρ is a weak solution of the partial differential equation (2.4).

Let $H \in C^2(\mathbb{T})$. We claim that

$$\mathbb{Q}^* \left\{ \pi : \langle \pi_t, H \rangle - \langle \pi_0, H \rangle - \int_0^t \langle \pi_s, \partial_u^2 H \rangle \, ds = 0, \, \forall t \in [0, T] \right\} = 1. \tag{3.8}$$

To prove this, it suffices to show that

$$\mathbb{Q}^* \left\{ \pi : \sup_{0 \leq t \leq T} \left| \langle \pi_t, H \rangle - \langle \pi_0, H \rangle - \int_0^t \langle \pi_s, \partial_u^2 H \rangle \, ds \right| > \delta \right\} = 0,$$

for every $\delta > 0$. Since the supremum is a continuous function in the Skorohod metric, by Portmanteau's Theorem, the probability above is smaller or equal than

$$\liminf_{n \to \infty} \mathbb{Q}^n_{\mu_n} \left\{ \pi : \sup_{0 \leq t \leq T} \left| \langle \pi_t, H \rangle - \langle \pi_0, H \rangle - \int_0^t \langle \pi_s, \partial_u^2 H \rangle \, ds \right| > \delta \right\}.$$

Since $\mathbb{Q}^n_{\mu_n}$ is the measure on the space $\mathcal{D}([0, T], \mathcal{M})$ induced by $\mathbb{P}^n_{\mu_n}$ via the empirical measure, we can rewrite the expression above as

$$\liminf_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \eta. : \sup_{0 \leq t \leq T} \left| \langle \pi^n_t, H \rangle - \langle \pi^n_0, H \rangle - \int_0^t \langle \pi^n_s, \partial^2_u H \rangle \, ds \right| > \delta \right\}.$$

Adding and subtracting $n^2 \mathfrak{L}_n \langle \pi^n_s, H \rangle$ to the integral term above, we can see that the previous expression is bounded from above by the sum of

$$\limsup_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \sup_{0 \leq t \leq T} \left| \langle \pi^n_t, H \rangle - \langle \pi^n_0, H \rangle - \int_0^t n^2 \mathfrak{L}_n \langle \pi^n_s, H \rangle \, ds \right| > \delta/2 \right\} \tag{3.9}$$

and

$$\limsup_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \left(n^2 \mathfrak{L}_n \langle \pi^n_s, H \rangle - \langle \pi^n_s, \partial^2_u H \rangle \right) ds \right| > \delta/2 \right\}. \tag{3.10}$$

As already verified in Section 3.1, the quadratic variation of the martingale $M^n_t(H)$ given in (3.2) goes to zero, as $n \to \infty$. Therefore, by Doob's inequality, expression (3.9) is null.

It remains to show that (3.10) also vanishes. Recall (3.6) for $n^2 \mathfrak{L}_n \langle \pi^n_s, H \rangle$. Let us examine its terms.

The first term in the sum (3.6) is

$$n \sum_{x \in \mathbb{T}_n \setminus \{0\}} \left(H\left(\frac{x+1}{n}\right) + H\left(\frac{x-1}{n}\right) - 2H\left(\frac{x}{n}\right) \right) \eta_s(x),$$

from which one can subtract

$$\langle \pi^n_s, \partial^2_u H \rangle = \frac{1}{n} \sum_{x \in \mathbb{T}_n} \partial^2_u H\left(\frac{x}{n}\right) \eta_s(x),$$

this difference being bounded (in modulus) by c_H/n, again because $H \in C^2(\mathbb{T})$, where $c_H > 0$ is a constant depending only on H.

The second term in (3.6) is

$$n\left(1 + o(1)\right)\left(H\left(\frac{1}{n}\right) + H\left(\frac{-1}{n}\right) - 2H\left(\frac{0}{n}\right) \right) \eta_s(0),$$

which converges to zero as $n \to \infty$, because $H \in C^2(\mathbb{T})$.

The last term in (3.6) is

$$n(1 - g(n))\left[\left(H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0) \, \eta_s(-1) \right.$$
$$\left. + \left(H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0) \, \eta_s(1) \right],$$

which goes to zero, as n goes to infinity, because H is smooth and $g(n) = 1 + o(1)$. By the facts above we conclude that (3.10) is zero, proving the claim.

Now, let $\{H_i\}_{i \geq 1}$ be a countable dense set of functions in $C^2(\mathbb{T})$, with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty + \|\partial^2_u H\|_\infty$. Intersecting a countable number of sets of probability one, (3.8) can be extended for all functions $H \in C^2(\mathbb{T})$ simultaneously, proving that $\mathbb{Q}^*$ is concentrated on weak solutions of (2.4). Since there exists only one weak solution of (2.4), it means that $\mathbb{Q}^*$

is equal to the aforementioned probability measure $\mathbb{Q}$. Invoking tightness proved in Section 3.1, we conclude that the entire sequence $\{\mathbb{Q}^n_{\mu_n}\}_{n \in \mathbb{N}}$ converges to $\mathbb{Q}$ as $n \to \infty$. $\quad\square$

3.5. Law of large numbers for the occupation at the origin

Recall the definition of μ_n given in (2.8).

Proposition 3.4. *Consider* $g(n) = \alpha n^{-\beta}$. *Let* $\gamma : \mathbb{T} \to [0, 1]$ *be a continuous profile, except possibly at* $x = 0$ *and satisfying* (2.7). *Then, for all* $t > 0$ *and* $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \eta. \ : \ \frac{n}{t} \int_0^t (1 - \eta_s(0)) \, ds \ > \ \varepsilon \right\} = 0. \tag{3.11}$$

This statement says that the site $x = 0$ remains empty a fraction of time smaller than $1/n$. A simple heuristics for this statement is the following. The time a particles takes to escape the slow site is, at least, an exponential random variable of parameter $g(n)$. If a random variable has exponential distribution of parameter λ, its expectation is $1/\lambda$. Hence, the time average that a trapped particle takes to escape from the slow site is at least n^β / α (if its neighboring sites are occupied, the trapped particle can spend even more time there). As time goes by, the slow site will remain empty a fraction of time at most $g(n) = \alpha n^{-\beta}$. Since $\beta > 1$, this would lead to (3.11).

Despite the simplicity of the heuristics above, it is not straightforward to transform it into a rigorous argument. In order to prove Proposition 3.4, we will make use of attractiveness and the knowledge on the invariant measures. For that purpose, in $\{0, 1\}^{\mathbb{T}_n}$ we introduce the natural order between configurations: $\eta \leq \zeta$ if and only if $\eta(x) \leq \zeta(x)$ for all $x \in \mathbb{T}_n$. A function $f : \{0, 1\}^{\mathbb{T}_n} \to \mathbb{R}$ is said to be monotone if $f(\eta) \leq f(\zeta)$ whenever $\eta \leq \zeta$. This partial order is naturally extended to the space of measures.

We write $\mu_1 \leq_{\text{st}} \mu_2$ if, and only if,

$$\int f \, d\mu_1 \ \leq \ \int f \, d\mu_2$$

for all monotone functions f. In this case, we say that μ_1 is *stochastically dominated from above* by μ_2. The next result is well known and can found in [15] for instance.

Theorem 3.5. *Let* μ_1 *and* μ_2 *be two probability measures on* $\{0, 1\}^{\mathbb{T}_n}$. *The statements below are equivalent:*

(1) $\mu_1 \leq_{\text{st}} \mu_2$;
(2) *There exists a probability measure* $\bar{\mu}$ *on* $\{0, 1\}^{\mathbb{T}_n} \times \{0, 1\}^{\mathbb{T}_n}$ *such that its first and second marginals are* μ_1 *and* μ_2, *respectively, and* $\bar{\mu}$ *is "concentrated above the diagonal", which means*

$$\bar{\mu} \left\{ (\eta, \zeta) \ : \ \eta \leq \zeta \right\} = 1.$$

Next, we construct such a measure $\bar{\mu}$, with the aforementioned property, by means of the so-called *graphical construction*.

Fix $n \in \mathbb{N}$. For each site x of $\mathbb{T}_n$, we associate two Poisson point processes $\mathcal{N}^{n,-}_x$ and $\mathcal{N}^{n,+}_x$, all of them being independent. The parameters of those Poisson process agree with the Fig. 1. In

other words, the parameter of $\mathcal{N}_x^{n,-}$ and of $\mathcal{N}_x^{n,+}$ is one for all $x \in \mathbb{T}_n$, except for $x = 0$, for which the parameter of $\mathcal{N}_0^{n,-}$ and $\mathcal{N}_0^{n,+}$ is equal to $g(n)$.

Given an initial configuration of particles $\eta \in \{0, 1\}^{\mathbb{T}_n}$ and the "toss" of those Poisson processes, the dynamics will be the following. At a time arrival of some Poisson process, let us say, a time arrival of $\mathcal{N}_x^{n,-}$, if there is a particle at the site x, and there is no particle at the site $x - 1$, the particle at x moves to $x - 1$. The analogous happens with respect to a Poisson process of type $\mathcal{N}_x^{n,+}$, in which the movement (if possible) is from x to $x + 1$. This construction yields the same Markov process previously defined via the generator given in (2.1).

Consider now two probability measures μ_1 and μ_2 in $\{0, 1\}^{\mathbb{T}_n}$ such that $\mu_1 \leq_{\text{st}} \mu_2$. By Theorem 3.5, there exists a measure $\bar{\mu}$ on $\{0, 1\}^{\mathbb{T}_n} \times \{0, 1\}^{\mathbb{T}_n}$ concentrated above the diagonal, as it is stated there. Evolving a configuration (η^1, η^2), chosen by $\bar{\mu}$, by the same set of Poisson point processes described above, we are lead to $\eta_t^1 \leq \eta_t^2$, for any future time $t > 0$. A stochastic process enjoying this property of preserving the partial order is said to be *attractive*. See Liggett's book [15] for more details on the subject.

Notice that the specific value of $g(n)$ does not play any role in the argument above. In resume, we can say that the defect does not destroy attractiveness.

Having established attractiveness of the exclusion process with a slow site we make the following observation. Once Theorem 2.2 is true for initial measures μ_n *conditioned to have a particle at the origin*, the statement will remain in force for initial measures μ_n. This is explained as follows.

By attractiveness, we can construct both processes (the one starting from μ_n and the one starting from μ_n conditioned to have a particle at the origin) in such a way that these processes will differ at most at one site, for any later time t. Therefore, the empirical measures (3.1) for each process will have the same limit in distribution.

Without loss of generality, we assume henceforth that there is a particle at the origin at the initial time.

Proof of Proposition 3.4. Since we have assumed $\gamma(x) \geq \zeta > 0$, for all $x \in \mathbb{T}$, since there is a particle at the origin and since μ_n is a product measure, we can find $p > 0$ small enough such that $\mu_n \geq_{\text{st}} \nu_p$, for any $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. By attractiveness,

$$\mathbb{P}_{\mu_n}\left\{ \eta_\cdot : \frac{n}{t} \int_0^t \eta_s(0) \, ds > \varepsilon \right\} \geq \mathbb{P}_p\left\{ \eta_\cdot : \frac{n}{t} \int_0^t \eta_s(0) \, ds > \varepsilon \right\},$$

which in turn implies

$$\mathbb{P}_{\mu_n}\left\{ \eta_\cdot : \frac{n}{t} \int_0^t (1 - \eta_s(0)) \, ds > \varepsilon \right\} \leq \mathbb{P}_p\left\{ \eta_\cdot : \frac{n}{t} \int_0^t (1 - \eta_s(0)) \, ds > \varepsilon \right\}.$$

By Chebyshev's inequality and Fubini's Theorem,

$$\mathbb{P}_p\left\{ \eta_\cdot : \frac{n}{t} \int_0^t (1 - \eta_s(0)) \, ds > \varepsilon \right\} \leq \frac{n}{\varepsilon} \mathbb{E}_p\left[\frac{1}{t} \int_0^t (1 - \eta_s(0)) \, ds \right]$$

$$= \frac{n}{\varepsilon}\left(1 - \frac{1}{t} \int_0^t \mathbb{E}_p[\eta_s(0)] \, ds \right).$$

Since

$$\nu_p\{\eta \, ; \, \eta(0) = 1\} = \frac{\frac{p}{g(n)}}{(1 - p) + \frac{p}{g(n)}},$$

we obtain that

$$\mathbb{P}_{\mu_n}\left\{\eta. \ : \ \frac{n}{t}\int_0^t (1 - \eta_s(0))\, ds > \varepsilon\right\} \leq \frac{n}{\varepsilon}\left(1 - \frac{\frac{p}{g(n)}}{(1 - p) + \frac{p}{g(n)}}\right),$$

finishing the proof since $g(n) = \alpha n^{-\beta}$, $\beta > 1$. $\quad\square$

3.6. Hydrodynamic limit for $g(n) = \alpha n^{-\beta}$

Recall that we have denoted $\lfloor \varepsilon n \rfloor$, the integer part of εn, simply by εn. Define the right average at $1 \in \mathbb{T}_n$, and the left average at $-1 \in \mathbb{T}_n$ by

$$\eta^{\varepsilon n, \mathrm{R}}(1) := \frac{1}{\varepsilon n}\sum_{y=1}^{\varepsilon n} \eta(y) \quad \text{and} \quad \eta^{\varepsilon n, \mathrm{L}}(-1) := \frac{1}{\varepsilon n}\sum_{y=n-\varepsilon n}^{n-1} \eta(y), \tag{3.12}$$

respectively. Notice that none of these sums involve the occupation at the slow site $0 \in \mathbb{T}_n$.

Proposition 3.6. *For any $t > 0$,*

$$\limsup_{\varepsilon \downarrow 0} \ \limsup_{n \to \infty} \ \mathbb{E}_{\mu_n}\left[\left|\int_0^t \left(\eta_s(1) - \eta_s^{\varepsilon n, \mathrm{R}}(1)\right) ds\right|\right] = 0$$

and

$$\limsup_{\varepsilon \downarrow 0} \ \limsup_{n \to \infty} \ \mathbb{E}_{\mu_n}\left[\left|\int_0^t \left(\eta_s(-1) - \eta_s^{\varepsilon n, \mathrm{L}}(-1)\right) ds\right|\right] = 0.$$

The last result says that we can replace the occupation at the neighboring sites of the slow site by their averages in closed boxes, provided that these boxes do not cross the slow site. This kind of argument appears often in the literature and can be found, for example, in [14].

Proof. We treat the case $x = +1$, the case $x = -1$ being analogous. From Jensen's inequality and the definition of the entropy, for any $N > 0$, the expectation appearing in the statement of this proposition is bounded from above by

$$\frac{\mathbf{H}(\mu_n|\nu_p)}{Nn} + \frac{1}{Nn}\log \mathbb{E}_p\left[\exp\left\{N n \left|\int_0^t \{\eta_s(1) - \eta_s^{\varepsilon n, \mathrm{R}}(1)\}\, ds\right|\right\}\right]. \tag{3.13}$$

By Proposition 3.2, $\mathbf{H}(\mu_n|\nu_p) \leq K_0\, n$, hence the term on the left hand side of last expression is bounded from above by K_0/N. Now, we bound the remaining term. Since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_n \frac{1}{n}\log(a_n + b_n) = \max\left\{\limsup_n \frac{1}{n}\log(a_n), \ \limsup_n \frac{1}{n}\log(b_n)\right\}, \tag{3.14}$$

we can remove the modulus inside the exponential. Moreover, by the Feynman–Kac formula[3] the term on the right hand side of (3.13) is less than or equal to

$$t \sup_{f \text{ density}}\left\{\int \{\eta(1) - \eta_s^{\varepsilon n, \mathrm{R}}(1)\} f(\eta)\nu_p(d\eta) - n\, \mathfrak{D}_n(\sqrt{f})\right\} ds.$$

---

[3] See, for example, Lemma A1.7.2 of [13].

Notice that the expression above does not depend on N. We claim now that, for any density f,

$$\int \{\eta(1) - \eta_s^{\varepsilon n,\,\text{R}}(1)\} f(\eta) \nu_p(d\eta) \le 2\varepsilon + n\, \mathfrak{D}_n(\sqrt{f}).$$

Since N is arbitrary large, once we prove this claim the proof will be finished. By the definition in (3.12),

$$\int \{\eta(1) - \eta_s^{\varepsilon n,\,\text{R}}(1)\} f(\eta) \nu_p(d\eta) = \int \left\{ \frac{1}{\varepsilon n} \sum_{y=1}^{\varepsilon n} (\eta(1) - \eta(y)) \right\} f(\eta)\, \nu_p(d\eta).$$

Writing $\eta(x) - \eta(y)$ as a telescopic sum, the right hand side of above can be rewritten as

$$\int \left\{ \frac{1}{\varepsilon n} \sum_{y=1}^{\varepsilon n} \sum_{z=1}^{y-1} (\eta(z) - \eta(z+1)) \right\} f(\eta)\, \nu_p(d\eta).$$

Rewriting the expression above as twice the half and making the transformation $\eta \mapsto \eta^{z,z+1}$ (for which the probability ν_p is invariant) it becomes

$$\frac{1}{2\varepsilon n} \sum_{y=1}^{\varepsilon n} \sum_{z=1}^{y-1} \int \{\eta(z) - \eta(z+1)\}(f(\eta) - f(\eta^{z,z+1}))\, \nu_p(d\eta).$$

By $(a - b) = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and Cauchy–Schwarz's inequality, we bound the previous expression from above by

$$\frac{1}{2\varepsilon n} \sum_{y=1}^{\varepsilon n} \sum_{z=1}^{y-1} A \int \{\eta(z) - \eta(z+1)\}^2 \left(\sqrt{f(\eta)} + \sqrt{f(\eta^{z,z+1})} \right)^2 \nu_p(d\eta)$$

$$+ \frac{1}{2\varepsilon n} \sum_{y=1}^{\varepsilon n} \sum_{z=1}^{y-1} \frac{1}{A} \int \left(\sqrt{f(\eta)} - \sqrt{f(\eta^{z,z+1})} \right)^2 \nu_p(d\eta),$$

for any $A > 0$. Since f is a density and recalling (3.7), the expression above is bounded by $A^{-1}\mathfrak{D}_n(f) + 2A\varepsilon n$. Choosing $A = 1/n$ we achieve the claim, concluding the proof. $\quad\square$

Proof of Proposition 3.1 for $g(n) = \alpha n^{-\beta}, \alpha > 0, \beta > 1$. Again, let $\mathbb{Q}^*$ be the weak limit of some convergent subsequence $\{\mathbb{Q}_{\mu_{n_j}}^{n_j}\}_{j \in \mathbb{N}}$ of the sequence $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ and to keep notation simple, denote this subsequence by $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$. Recall that by Proposition 3.3, the probability measure $\mathbb{Q}^*$ is concentrated on trajectories $\pi(t, du) = \rho(t, u)du$ such that $\rho(t, u) \in L^2(0, T; \mathcal{H}^1)$. By the notion of trace in Sobolev spaces, the integrals

$$\int_0^t \rho(s, 0)\, ds \quad \text{and} \quad \int_0^t \rho(s, 1)\, ds \tag{3.15}$$

are well defined and are finite. See [5] for the properties of Sobolev spaces. More than that, since the Sobolev space in one dimension is composed of functions which are absolutely continuous, ρ has indeed left and right limits at zero.

Our goal here is to conclude that ρ is a weak solution of the partial differential equation (2.6).

For this purpose, let $H \in C^2[0, 1]$. Notice that, if H is seen as a function in the torus, H is possibly discontinuous at zero. We impose that $H(0) = 0$. We claim that

$$\mathbb{Q}^* \Big\{ \pi \; : \; \langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t \big(\rho_s(0) \, \partial_u H(0) - \rho_s(1) \, \partial_u H(1) \big) \, ds = 0, \; \forall t \in [0, T] \Big\} = 1. \tag{3.16}$$

To prove this, it suffices to show that

$$\mathbb{Q}^* \Big\{ \pi \; : \; \sup_{0 \le t \le T} \Big| \langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t \big(\rho_s(0) \, \partial_u H(0) - \rho_s(1) \, \partial_u H(1) \big) \, ds \Big| > \delta \Big\} = 0,$$

for any $\delta > 0$. Since the integrals in (3.15) are not defined in the whole Skorohod space $\mathcal{D}([0, T], \mathcal{M})$, we cannot apply Portmanteau's Theorem yet.

For that purpose, let $\iota_\varepsilon(u) = \varepsilon^{-1} \mathbf{1}_{(0, \varepsilon]}(u)$. Adding and subtracting the convolution of $\rho(t, u)$ with ι_ε at the boundaries, we bound the previous probability by the sum of

$$\mathbb{Q}^* \Big\{ \pi \; : \; \sup_{0 \le t \le T} \Big| \langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t (\rho_s * \iota_\varepsilon)(0) \, \partial_u H(0) \, ds + \int_0^t (\rho_s * \iota_\varepsilon)(1 - \varepsilon) \, \partial_u H(1) \, ds \Big| > \delta/2 \Big\} \tag{3.17}$$

and

$$\mathbb{Q}^* \Big\{ \pi \; : \; \sup_{0 \le t \le T} \Big| \int_0^t (\rho_s * \iota_\varepsilon)(0) \, \partial_u H(0) \, ds - \int_0^t (\rho_s * \iota_\varepsilon)(1 - \varepsilon) \, \partial_u H(1) \, ds$$

$$- \int_0^t \big(\rho_s(0) \, \partial_u H(0) + \int_0^t \rho_s(1) \, \partial_u H(1) \big) \, ds \Big| > \delta/2 \Big\}.$$

Since ρ has left and right side limits, taking ε small, the previous probability goes to zero, as $n \to \infty$. It remains to bound (3.17). By Portmanteau's Theorem and since there is at most one particle per site, (3.17) is bounded from above by

$$\limsup_{n \to \infty} \mathbb{Q}_{\mu_n}^n \Big\{ \pi \; : \; \sup_{0 \le t \le T} \Big| \langle \pi_t^n, H \rangle - \langle \pi_0^n, H \rangle - \int_0^t \langle \pi_s^n, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t \langle \pi_t^n, \varepsilon^{-1} \mathbf{1}_{(0, \varepsilon]} \rangle \, \partial_u H(0) \, ds + \int_0^t \langle \pi_t^n, \varepsilon^{-1} \mathbf{1}_{(1-\varepsilon, 1]} \rangle \, \partial_u H(1) \, ds \Big| > \delta/2 \Big\}. \tag{3.18}$$

Noticing the identities

$$\eta_s^{\varepsilon n, \mathrm{R}}(1) = \langle \pi_s^n, \varepsilon^{-1} \mathbf{1}_{(0, \varepsilon]} \rangle \quad \text{and} \quad \eta_s^{\varepsilon n, \mathrm{L}}(-1) = \langle \pi_s^n, \varepsilon^{-1} \mathbf{1}_{(1-\varepsilon, 1]} \rangle,$$

we can rewrite (3.18) as

$$\limsup_{n \to \infty} \mathbb{P}_{\mu_n} \Big\{ \eta \; : \; \sup_{0 \le t \le T} \Big| \langle \pi_t^n, H \rangle - \langle \pi_0^n, H \rangle - \int_0^t \langle \pi_s^n, \partial_u^2 H \rangle \, ds$$

$$- \int_0^t \eta_s^{\varepsilon n, \mathrm{R}}(1) \, \partial_u H(0) \, ds + \int_0^t \eta_s^{\varepsilon n, \mathrm{L}}(-1) \, \partial_u H(1) \, ds \Big| > \delta/2 \Big\}.$$

Recalling Proposition 3.6, in order to prove that the limit above is equal to zero, it is enough to show that the limit below is null:

$$\limsup_{n\to\infty} \mathbb{P}_{\mu_n}\Big\{\eta: \sup_{0\le t\le T}\Big|\langle\pi_t^n, H\rangle - \langle\pi_0^n, H\rangle - \int_0^t \langle\pi_s^n, \partial_u^2 H\rangle\, ds$$

$$- \int_0^t \eta_s(1)\,\partial_u H(0)\, ds + \int_0^t \eta_s(-1)\,\partial_u H(1)\, ds\Big| > \delta/2\Big\}.$$

Adding and subtracting $n^2 \mathfrak{L}_n \langle\pi_s^n, H\rangle$, the previous expression is bounded from above by the sum of

$$\limsup_{n\to\infty} \mathbb{P}_{\mu_n}\Big\{\sup_{0\le t\le T}\Big|\langle\pi_t^n, H\rangle - \langle\pi_0^n, H\rangle - \int_0^t n^2 \mathfrak{L}_n \langle\pi_s^n, H\rangle\, ds\Big| > \delta/4\Big\} \tag{3.19}$$

and

$$\limsup_{n\to\infty} \mathbb{P}_{\mu_n}\Big\{\sup_{0\le t\le T}\Big|\int_0^t n^2 \mathfrak{L}_n \langle\pi_s^n, H\rangle\, ds - \int_0^t \langle\pi_s^n, \partial_u^2 H\rangle\, ds$$

$$- \int_0^t \eta_s(1)\,\partial_u H(0)\, ds + \int_0^t \eta_s(-1)\,\partial_u H(1)\, ds\Big| > \delta/4\Big\}. \tag{3.20}$$

As can be easily verified, by the imposed conditions on the test function H, the quadratic variation of the martingale $M_t^n(H)$ given in (3.2) goes to zero, as $n\to\infty$. Therefore, by Doob's inequality, (3.19) is null.

It remains to show that (3.20) also vanishes. Recall (3.6) for $n^2 \mathfrak{L}_n \langle\pi_s^n, H\rangle$. Let us examine its terms. The first term in the sum (3.6) is

$$n \sum_{x\in\mathbb{T}_n\setminus\{0\}} \Big(H\Big(\frac{x+1}{n}\Big) + H\Big(\frac{x-1}{n}\Big) - 2H\Big(\frac{x}{n}\Big)\Big)\eta_s(x),$$

which we split into the sum of

$$n \sum_{x\in\mathbb{T}_n\setminus\{-1,0,1\}} \Big(H\Big(\frac{x+1}{n}\Big) + H\Big(\frac{x-1}{n}\Big) - 2H\Big(\frac{x}{n}\Big)\Big)\eta_s(x) \tag{3.21}$$

and

$$n\Big(H\Big(\frac{2}{n}\Big) - 2H\Big(\frac{1}{n}\Big)\Big)\eta_s(1) + n\Big(H\Big(\frac{-1}{n}\Big) - 2H\Big(\frac{-2}{n}\Big)\Big)\eta_s(-1). \tag{3.22}$$

The difference between (3.21) and

$$\langle\pi_s^n, \partial_u^2 H\rangle = \frac{1}{n}\sum_{x\in\mathbb{T}_n} \partial_u^2 H\Big(\frac{x}{n}\Big)\,\eta_s(x)$$

in bounded (in modulus) by c_H/n, because $H\in C^2[0,1]$, where $c_H > 0$ is a constant depending only on H. The second term in the sum (3.6) is

$$n^{1-\beta}\Big(H\Big(\frac{1}{n}\Big) + H\Big(\frac{-1}{n}\Big) - 2H\Big(\frac{0}{n}\Big)\Big)\eta_s(0)$$

which converges to zero as $n \to \infty$, because $\beta > 1$. The last term in (3.6) is

$$
n(1 - g(n)) \left[\left(H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0)\, \eta_s(-1) \right.
$$
$$
\left. + \left(H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0)\, \eta_s(1) \right],
$$

which can be rewritten as the sum of

$$
n(1 - g(n)) \left[\left(H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(-1) + \left(H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(1) \right] \tag{3.23}
$$

and

$$
n(\eta_s(0) - 1)(1 - g(n)) \left[\left(H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(-1) \right.
$$
$$
\left. + \left(H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(1) \right].
$$

By Proposition 3.4, the time integral of the last term in the previous expression converges to zero in probability, as $n \to \infty$. Since $H\left(\frac{0}{n}\right) = 0$ and $\beta > 1$, the expression (3.22) plus the expression (3.23) is equal to

$$
n\left(H\left(\frac{2}{n}\right) - H\left(\frac{1}{n}\right) \right) \eta_s(1) + n\left(H\left(\frac{-1}{n}\right) - H\left(\frac{-2}{n}\right) \right) \eta_s(-1),
$$

plus an error of order $n^{1-\beta}$. The expression above is, asymptotically in n, the same as

$$
\eta_s(1)\, \partial_u H(0) - \eta_s(-1)\, \partial_u H(1),
$$

whose time integral cancels with the remaining time integrals of (3.20) and therefore proves that (3.20) vanishes. This concludes the claim (3.16).

Now, let $\{H_i\}_{i \geq 1}$ be a countable dense set of functions on $C^2[0, 1]$, with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty + \|\partial_u^2 H\|_\infty$. Since (3.16) is true for each one of these functions H_i, we can extend (3.16) for all functions $H \in C^2[0, 1]$ simultaneously by intersecting a countable number of sets of probability one. This proves that $\mathbb{Q}^*$ is concentrated on weak solutions of (2.6). Since there exists only one weak solution of (2.6), it means that $\mathbb{Q}^*$ is equal to the aforementioned probability measure $\mathbb{Q}$. Invoking the tightness that we have proved in Section 3.1, we conclude that the entire sequence $\{\mathbb{Q}_{\mu_n}^n\}_{n \in \mathbb{N}}$ converges to $\mathbb{Q}$, as $n \to \infty$. $\quad\square$

4. Equilibrium density fluctuations

In this section we prove Theorem 2.10. Recall that here we take the process evolving on $\mathbb{Z}$ and recall also (2.10). By Dynkin's formula,

$$
\mathcal{M}_t^n(H) := \mathcal{Y}_t^n(H) - \mathcal{Y}_0^n(H) - \int_0^t n^2 \mathfrak{L}_n \mathcal{Y}_s^n(H)\, ds, \tag{4.1}
$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \leq t)$. Besides that,

$$
\left(\mathcal{M}_t^n(H) \right)^2 - \int_0^t \left(n^2 \mathfrak{L}_n \left(\mathcal{Y}_t^n(H) \right)^2 - 2\, \mathcal{Y}_s^n(H)\, n^2 \mathfrak{L}_n\, \mathcal{Y}_s^n(H) \right) ds, \tag{4.2}
$$

is also a martingale with respect to the same filtration. By

$$\mathcal{I}_t^n(H) = \int_0^t n^2 \mathfrak{L}_n \mathcal{Y}_s^n(H) \, ds$$

we denote the integral part in (4.1). First we observe that the expression

$$\sum_{x \in \mathbb{Z}} \left(H\left(\frac{x+1}{n}\right) + H\left(\frac{x-1}{n}\right) - 2H\left(\frac{x}{n}\right) \right) p$$

is well defined since H decays fast and is equal to zero. Analogously to (3.6), some direct calculations then yield

$$n^2 \mathfrak{L}_n \mathcal{Y}_s^n(H) = n^{3/2} \sum_{x \neq -1,0,1} \left[H\left(\frac{x+1}{n}\right) + H\left(\frac{x-1}{n}\right) - 2H\left(\frac{x}{n}\right) \right] \bar{\eta}_s(x)$$

$$+ n^{3/2} \left[H\left(\frac{2}{n}\right) + H\left(\frac{0}{n}\right) - 2H\left(\frac{1}{n}\right) \right] \bar{\eta}_s(1)$$

$$+ n^{3/2} \left[H\left(\frac{-2}{n}\right) + H\left(\frac{0}{n}\right) - 2H\left(\frac{-1}{n}\right) \right] \bar{\eta}_s(-1)$$

$$+ n^{3/2}(1 - g(n)) \left[\left(H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0)\eta_s(1) \right.$$

$$+ \left. \left(H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right) \right) \eta_s(0)\eta_s(-1) \right]$$

$$+ n^{3/2} g(n) \left[H\left(\frac{1}{n}\right) + H\left(\frac{-1}{n}\right) - 2H\left(\frac{0}{n}\right) \right] \eta_s(0) + \Theta(n, p, H), \qquad (4.3)$$

where $\bar{\eta}_s(x) = \eta(x) - m_p(x)$ is the centered occupation variable, as in (2.10) and

$$\Theta(n, p, H) = -n^{3/2} \left[H\left(\frac{1}{n}\right) + H\left(\frac{-1}{n}\right) - 2H\left(\frac{0}{n}\right) \right] p.$$

The next two propositions are direct calculations very similar to [8], and for this reason their proofs are omitted.

Proposition 4.1. *Consider* $g(n) = 1 + o(1)$ *and* $H \in \mathcal{S}(\mathbb{R})$. *In this case,*

$$\lim_{n \to \infty} \mathbb{E}_p \left[(\mathcal{M}_t^n(H))^2 \right] = 2t \, \chi(p) \|\nabla H\|_{L^2(\mathbb{R})}^2$$

and

$$\limsup_{n \to \infty} \mathbb{E}_p \left[(\mathcal{I}_t^n(H))^2 \right] \leq 80 \, t \, \chi(p) \|\nabla H\|_{L^2(\mathbb{R})}^2.$$

Proposition 4.2. *Consider* $g(n) = \alpha n^{-\beta}$, $\alpha > 0$, $\beta > 1$, *and* $H \in \mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$. *In this case,*

$$\lim_{n \to \infty} \mathbb{E}_p \left[(\mathcal{M}_t^n(H))^2 \right] = 2t \, \chi(p) \|\nabla_{\mathrm{Neu}} H\|_{L^2(\mathbb{R})}^2$$

and

$$\limsup_{n \to \infty} \mathbb{E}_p \left[(\mathcal{I}_t^n(H))^2 \right] \leq 80 \, t \, \chi(p) \|\nabla_{\mathrm{Neu}} H\|_{L^2(\mathbb{R})}^2.$$

The next result is concerned with convergence at initial time, for either case of the function $g(n)$.

Proposition 4.3. $\{\mathcal{Y}_0^n\}_{n \in \mathbb{N}}$ *converges in distribution to* $\mathcal{Y}_0$, *as* $n \to \infty$, *where* $\mathcal{Y}_0$ *is a Gaussian field with mean zero and covariance given by* (2.14).

Mutatis mutandis, the same proof of [8, Prop. 3.2] applies and is suppressed here.

4.1. Tightness

Here we prove tightness of the process $\{\mathcal{Y}_t^n; t \in [0, T]\}_{n \in \mathbb{N}}$ in both cases of $g(n)$. First we notice that by Mitoma's criterion [16] and the fact that $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ are Fréchet spaces, it is enough to prove tightness of the sequence of real-valued processes $\{\mathcal{Y}_t^n(H); t \in [0, T]\}_{n \in \mathbb{N}}$, where $H \in \mathcal{S}(\mathbb{R})$ if we consider $g(n) = 1 + o(1)$, and $H \in \mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ if we consider $g(n) = \alpha n^{-\beta}$, $\alpha > 0, \beta > 1$.

In order to prove tightness of $\{\mathcal{Y}_t^n(H) : 0 \le t \le T\}_{n \in \mathbb{N}}$, we shall prove tightness of each term in the formula (4.1).

Fix a test function H belonging to the respective space for each case of $g(n)$. By (4.1), it is enough to prove tightness of the stochastic processes $\{\mathcal{Y}_0^n(H)\}_{n \in \mathbb{N}}$, $\{\mathcal{I}_t^n(H); t \in [0, T]\}_{n \in \mathbb{N}}$, and $\{\mathcal{M}_t^n(H); t \in [0, T]\}_{n \in \mathbb{N}}$.

By Proposition 4.3 we have convergence at initial time, hence $\{\mathcal{Y}_0^n(H)\}_{n \in \mathbb{N}}$ is obviously tight. To show tightness of the remaining real-valued processes we use the Aldous criterion:

Proposition 4.4 (*Aldous' Criterion*). *A sequence* $\{x_t^n; t \in [0, T]\}_{n \in \mathbb{N}}$ *of real-valued processes is tight with respect to the Skorohod topology of* $\mathcal{D}([0, T], \mathbb{R})$ *if:*

(i) $\lim_{A \to +\infty} \lim\sup_{n \to +\infty} \mathbb{P}\left(\sup_{0 \le t \le T} |x_t^n| > A\right) = 0$,

(ii) *for any* $\varepsilon > 0$, $\lim_{\delta \to 0} \lim\sup_{n \to +\infty} \sup_{\lambda \le \delta} \sup_{\tau \in \mathcal{T}_T} \mathbb{P}(|x_{\tau+\lambda}^n - x_\tau^n| > \varepsilon) = 0$,

where $\mathcal{T}_T$ *is the set of stopping times bounded by* T.

For the martingale term, the claim (i) of Aldous' criterion is achieved by an application of Doob's inequality together with Proposition 4.1 or Proposition 4.2 (depending on the chosen g).

By Proposition 4.1 or Proposition 4.2, the claim (i) of Aldous' criterion can be easily checked for the integral term. It remains to check (ii). Fix a stopping time $\tau \in \mathcal{T}_T$ and suppose that $g(n) = 1 + o(1)$. By Chebyshev's inequality,

$$\mathbb{P}_p\left(\left|\mathcal{M}_{\tau+\lambda}^n(H) - \mathcal{M}_\tau^n(H)\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2} \mathbb{E}_p\left[\left(\mathcal{M}_{\tau+\lambda}^n(H) - \mathcal{M}_\tau^n(H)\right)^2\right].$$

Thus, by Proposition 4.1,

$$\lim_{n \to \infty}\sup \mathbb{P}_p\left(\left|\mathcal{M}_{\tau+\lambda}^n(H) - \mathcal{M}_\tau^n(H)\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2} 2\chi(p)\lambda \|\nabla H\|_{L^2(\mathbb{R})}^2$$

$$\le \frac{1}{\varepsilon^2} 2\chi(p)\delta \|\nabla H\|_{L^2(\mathbb{R})}^2,$$

which vanishes as $\delta \to 0$. Similarly,

$$\mathbb{P}_p\left(\left|\mathcal{I}_{\tau+\lambda}^n(H) - \mathcal{I}_\tau^n(H)\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2} \mathbb{E}_p\left[\left(\mathcal{I}_{\tau+\lambda}^n(H) - \mathcal{I}_\tau^n(H)\right)^2\right].$$

Again by [Proposition 4.1](#), we obtain

$$\limsup_{n\to\infty} \mathbb{P}_p\big(\big|\mathcal{I}^n_{\tau+\lambda}(H) - \mathcal{I}^n_{\tau}(H)\big| > \varepsilon\big) \le \frac{80t}{\varepsilon^2}\delta\,\chi(p)\|\nabla H\|^2_{L^2(\mathbb{R})},$$

which vanishes as $\delta \to 0$.

The proof in the case $g(n) = \alpha n^{-\beta}$ is analogous (invoking is this case [Proposition 4.2](#)) and for this reason will be omitted. This finishes the proof of tightness.

4.2. Characterization of limit points for $g(n) = 1 + o(1)$

We shall prove that any limit of $\{\mathcal{Y}^n_t(H)\}_{n\in\mathbb{N}}$ is concentrated on solutions of the martingale problem described in [Proposition 2.8](#), with $H \in \mathcal{S}(\mathbb{R})$. Suppose that $\{\mathcal{Y}^n_t\}_{n\in\mathbb{N}}$ converges along a subsequence to $\mathcal{Y}_t$. In slight abuse of notation, we denote this convergent subsequence also by $\{\mathcal{Y}^n_t\}_{n\in\mathbb{N}}$.

In this case $H \in \mathcal{S}(\mathbb{R})$ is smooth, hence we have

$$\left| n^2\left[H\left(\frac{x+1}{n}\right) + H\left(\frac{x-1}{n}\right) - 2H\left(\frac{x}{n}\right)\right] - \Delta H(x)\right| \le \frac{c_H}{n}.$$

A similar analysis to the one presented in [(4.3)](#) for the hydrodynamic limit implies that

$$\mathcal{M}^n_t(H) := \mathcal{Y}^n_t(H) - \mathcal{Y}^n_0(H) - \int_0^t \mathcal{Y}^n_s(\Delta H)\,ds + e(n),$$

where the error function $e(n)$ is bounded, in modulus, by $cn^{-1/2}$. Since we are supposing that $\{\mathcal{Y}^n_t\}_{n\in\mathbb{N}}$ converges, we conclude that $\{\mathcal{M}^n_t(H)\}_{n\in\mathbb{N}}$ converges.

By similar arguments to those presented in [8], we know that the sequence of martingales $\{\mathcal{M}^n_t(H)\}_{n\in\mathbb{N}}$ is uniformly integrable. This implies that the limit of $\{\mathcal{M}^n_t(H)\}_{n\in\mathbb{N}}$, which we denote by $\mathcal{M}_t(H)$, is again a martingale. By [Proposition 4.1](#), its quadratic variation is $2\chi(p)\,t\,\|\nabla H\|^2_{L^2(\mathbb{R})}$. Now, [Proposition 2.8](#) finishes the characterization of limit points in this case.

4.3. Characterization of limit points for $g(n) = \alpha n^{-\beta}$

We shall prove in this case that any limit of $\{\mathcal{Y}^n_t(H)\}_{n\in\mathbb{N}}$ with $H \in \mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ is a solution of the martingale problem described in [Proposition 2.9](#).

In this situation, there is no analogous result of [Proposition 3.4](#). The key ingredient here will be the following tricky lemma:

Lemma 4.5. *Let* $g(n) = \alpha n^{-\beta}$ *and* $x = \pm 1$. *For some constant* $C > 0$ *not depending on* n *the estimates*

$$\mathbb{E}_p\left[\left(\int_0^t \left(n^{3/2}\eta_s(x)(1-\eta_s(0)) - \alpha n^{\frac{3}{2}-\beta}\eta_s(0)(1-\eta_s(x))\right)ds\right)^2\right] \le Cn^{1-\beta},$$

hold.

Proof. We prove only the inequality for $x = 1$, the case $x = -1$ is completely analogous. By the Kipnis–Varadhan inequality (see [13, Proposition A1.6.1]), the expectation

$$\mathbb{E}_p\left[\left(\int_0^t \left(n^{3/2}\eta_s(1)(1-\eta_s(0)) - \alpha n^{\frac{3}{2}-\beta}\eta_s(0)(1-\eta_s(1))\right)ds\right)^2\right]$$

is less or equal than

$$
t \sup_{f \in L^2(\nu_p)} \left\{ \int n^{3/2} \eta(1)(1 - \eta(0)) f(\eta) \nu_p(d\eta) \right.
$$

$$
\left. - \alpha n^{\frac{3}{2} - \beta} \int \eta(0)(1 - \eta(1)) f(\eta) \nu_p(d\eta) - n^2 \mathfrak{D}_n(f) \right\}
$$

where $\mathfrak{D}_n$ is the Dirichlet form[4] given in (3.7). In the first integral inside the supremum above we perform the change of variables $\eta \mapsto \eta^{1,0}$. Thus, the expression above can be rewritten as

$$
t \sup_{f \in L^2(\nu_p)} \left\{ \int n^{3/2} \eta(0)(1 - \eta(1)) f(\eta^{0,1}) \nu_p(d\eta^{0,1}) \right.
$$

$$
\left. - \alpha n^{\frac{3}{2} - \beta} \int \eta(0)(1 - \eta(1)) f(\eta) \nu_p(d\eta) - n^2 \mathfrak{D}_n(f) \right\}. \tag{4.4}
$$

Now, from (2.3) we have that

$$
\frac{\nu_p(\{\eta^{0,1}\})}{\nu_p(\{\eta\})} = \left(\frac{m_p(0)}{p} \right)^{\eta(1)} \left(\frac{p}{m_p(0)} \right)^{\eta(0)} \left(\frac{1-p}{1-m_p(0)} \right)^{1-\eta(0)} \left(\frac{1-m_p(0)}{1-p} \right)^{1-\eta(1)}
$$

from where we get

$$
\int n^{3/2} \eta(0)(1 - \eta(1)) f(\eta^{0,1}) \nu_p(d\eta^{0,1}) = \int \alpha n^{\frac{3}{2} - \beta} \eta(0)(1 - \eta(1)) f(\eta^{1,0}) \nu_p(d\eta),
$$

and therefore (4.4) is the same as

$$
t \sup_{f \in L^2(\nu_p)} \left\{ \alpha n^{\frac{3}{2} - \beta} \int \eta(0)(1 - \eta(1)) \left(f(\eta^{1,0}) - f(\eta) \right) \nu_p(d\eta) - n^2 \mathfrak{D}_n(f) \right\}.
$$

By the inequality $xy \leq \frac{Ax^2}{2} + \frac{y^2}{2A}$, $\forall A > 0$, the expression above is smaller than

$$
t \sup_{f \in L^2(\nu_p)} \left\{ \frac{A\alpha n^{\frac{3}{2} - \beta}}{2} \int \eta(0)(1 - \eta(1)) \left(f(\eta^{1,0}) - f(\eta) \right)^2 \nu_p(d\eta) \right.
$$

$$
\left. + \frac{\alpha n^{\frac{3}{2} - \beta}}{2A} \int \eta(0)(1 - \eta(1)) \nu_p(d\eta) - n^2 \mathfrak{D}_n(f) \right\}
$$

for any $A > 0$. Picking $A = \sqrt{n}$ the last expression becomes equal to

$$
t \sup_{f \in L^2(\nu_p)} \left\{ \frac{\alpha n^{2 - \beta}}{2} \int \eta(0)(1 - \eta(1)) \left(f(\eta^{1,0}) - f(\eta) \right)^2 \nu_p(d\eta) \right.
$$

$$
\left. + \frac{\alpha n^{1 - \beta}}{2} \int \eta(0)(1 - \eta(1)) \nu_p(d\eta) - n^2 \mathfrak{D}_n(f) \right\}.
$$

---

[4] Notice that here the Dirichlet form is evaluated at f instead of $\sqrt{f}$ as in Section 3.3.

Since the Dirichlet form (3.7) is a sum of positive terms, and since the first term above is exactly the first term in $n^2 \mathfrak{D}_n(f)$, we conclude that the expression above is less or equal than

$$\frac{tan^{1-\beta}}{2} \int \eta(0)(1 - \eta(1))\nu_p(d\eta) = Cn^{1-\beta},$$

for $C > 0$ not depending on n. This concludes the proof. □

Let us proceed to the characterization of limit points in this case. We begin with an observation that will strongly simplify the analysis.

First of all, we notice that $\mathcal{M}_t^n(H)$ defined in (4.1) is a martingale where $H \in \mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ does not play any special role, except the decay at infinity to make the sum well defined. In fact, we can take $H = H_n$ depending on n. We will do that in the following way. For each $n \in \mathbb{N}$, we impose $H_n\left(\frac{x}{n}\right) = H\left(\frac{x}{n}\right)$ for all $x \neq 0$ while for $x = 0$ we impose

$$H_n\left(\frac{0}{n}\right) = \frac{1}{2}\left(H\left(\frac{1}{n}\right) + H\left(\frac{-1}{n}\right)\right). \tag{4.5}$$

In this way, we obtain

$$H_n\left(\frac{1}{n}\right) + H_n\left(\frac{-1}{n}\right) - 2H_n\left(\frac{0}{n}\right) = 0, \quad \forall n \in \mathbb{N} \tag{4.6}$$

which cancels two parcels in (4.3). To simplify notation we will write H instead of H_n, keeping in mind (4.6).

We examine carefully all the terms in (4.3). By the discussion above, both $\Theta(n, p, H)$ and

$$n^{3/2}g(n)\left[H\left(\frac{1}{n}\right) + H\left(\frac{-1}{n}\right) - 2H\left(\frac{0}{n}\right)\right]\eta(0) \tag{4.7}$$

vanish. Let us see the remaining terms. The first sum on the right hand side of (4.3) is equal to

$$\frac{1}{\sqrt{n}} \sum_{x \neq -1,0,1} \Delta_{\mathrm{Neu}} H\left(\frac{x}{n}\right) \bar{\eta}_s(x)$$

plus an error of order $O(n^{-1/2})$. Since the side derivatives of $H \in \mathcal{S}_{\mathrm{Neu}}(\mathbb{R})$ at zero vanish, we also have that the second and third terms in (4.3) are equal to

$$n^{3/2}\left[H\left(\frac{0}{n}\right) - H\left(\frac{1}{n}\right)\right]\bar{\eta}_s(1) + n^{3/2}\left[H\left(\frac{0}{n}\right) - H\left(\frac{-1}{n}\right)\right]\bar{\eta}_s(-1)$$

plus another error of order $O(n^{-1/2})$. By (4.5), the expression above can be rewritten as

$$\frac{n^{3/2}}{2}\left[H\left(\frac{-1}{n}\right) - H\left(\frac{1}{n}\right)\right]\eta(1) + \frac{n^{3/2}}{2}\left[H\left(\frac{1}{n}\right) - H\left(\frac{-1}{n}\right)\right]\eta(-1).$$

Last expression together with the remaining two parcels in (4.3) gives us the sum of

$$\frac{n^{3/2}}{2}\left[H\left(\frac{-1}{n}\right) - H\left(\frac{1}{n}\right)\right]\eta(1) + \frac{n^{3/2}}{2}(1 - g(n))$$

$$\times \left[H\left(\frac{1}{n}\right) - H\left(\frac{-1}{n}\right)\right]\eta_s(0)\eta_s(1) \tag{4.8}$$

and

$$\frac{n^{3/2}}{2}\left[H\left(\frac{1}{n}\right) - H\left(\frac{-1}{n}\right)\right]\eta(-1) + \frac{n^{3/2}}{2}(1 - g(n))$$
$$\times \left[H\left(\frac{-1}{n}\right) - H\left(\frac{1}{n}\right)\right]\eta_s(0)\eta_s(-1). \tag{4.9}$$

At this point, we will use (4.7). Regardless of the fact that (4.7) is null, we can split it in two parts, namely $n^{3/2}g(n)\left[H\left(\frac{1}{n}\right) - H\left(\frac{0}{n}\right)\right]\eta(0)$ and $n^{3/2}g(n)\left[H\left(\frac{-1}{n}\right) - H\left(\frac{0}{n}\right)\right]\eta(0)$. The first one we add to (4.8) and the second one to (4.9). Recalling (4.5), it gives us the sum of

$$\frac{n^{3/2}}{2}\left(H\left(\frac{-1}{n}\right) - H\left(\frac{1}{n}\right)\right)\left[\eta_s(1)(1 - \eta_s(0)) - g(n)\eta_s(0)(1 - \eta_s(1))\right] \tag{4.10}$$

and

$$\frac{n^{3/2}}{2}\left(H\left(\frac{1}{n}\right) - H\left(\frac{-1}{n}\right)\right)\left[\eta_s(-1)(1 - \eta_s(0)) - g(n)\eta_s(0)(1 - \eta_s(-1))\right]. \tag{4.11}$$

Lemma 4.5 asserts that the time integrals of expressions (4.10) and (4.11) are asymptotically negligible in L^2.

For $H \in \mathcal{S}_{\text{Neu}}(\mathbb{R})$, the sequence of martingales $\{\mathcal{M}_t^n(H)\}_{n\in\mathbb{N}}$ presented in (4.1) is uniformly integrable. This implies that the L^2-limit of $\{\mathcal{M}_t^n(H)\}_{n\in\mathbb{N}}$, denoted by $\mathcal{M}_t(H)$, is again a martingale which quadratic variation given $2\chi(p)\,t\,\|\nabla_{\text{Neu}}H\|_{L^2(\mathbb{R})}^2$, assured by Proposition 4.1.

The entire previous discussion on the integral part of $\mathcal{M}_t^n(H)$ lead us to conclude that $\mathcal{M}_t(H)$ satisfies

$$\mathcal{M}_t(H) := \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\Delta_{\text{Neu}}H)\,ds,$$

which concludes the characterization of limit points by Proposition 2.9.

5. Open questions and conjectures

As presented in this paper, the critical defect strength and behavior at the critical point remains open in sense that it is not clear what should be the limit for $0 \le \beta \le 1$ when $g(n) = \alpha n^{-\beta}$, $\alpha > 0$. One conceivable scenario is that for any $\beta > 0$, both hydrodynamic limit and fluctuations would be driven by a disconnect behavior corresponding to Neumann boundary conditions, meaning that the critical point would be $\beta_c = 0$.

Our guess instead is that the correct critical point should be achieved at $\beta = 1$, much more close to the scenario of [6] where a *slow bond* is considered instead of a *slow site*. The physical intuition behind the dynamical phase transition taking place at $\beta = 1$ is the fact that, in a large but finite system and at large but finite times, the particle current, which is of diffusive origin, will be of order $1/n$ everywhere (with some space-dependent amplitude that depends on the initial state) before equilibrium is reached. However, a weak site with $\beta > 1$ cannot allow such a current to flow and hence it acts like a total blockage corresponding to Neumann boundary conditions. On the other hand, a defect rate with $\beta < 1$ does not make a current of order $1/n$ impossible, corresponding to a macroscopically irrelevant local perturbation of the particle system.
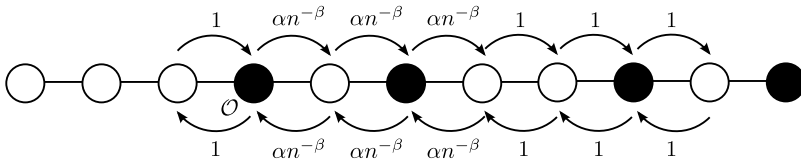
Fig. 2. Exclusion process with three neighboring *slow bonds*.

Specifically we conjecture that the behavior for $\beta = 1$ should be described by the partial differential equation

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in (0, 1), \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = \dfrac{\alpha}{2}(\rho(t, 0^+) - \rho(t, 0^-)), & t \geq 0, \\ \rho(0, u) = \rho_0(u), & u \in (0, 1), \end{cases} \tag{5.1}$$

where 0^+ and 0^- denotes right and left side limits, respectively. This conjecture is motivated by the observation that the equation above is the hydrodynamic equation of two neighboring slow bonds at $\beta = 1$, a claim made precise in next section. Thus our conjecture in the slow site setting is the following.

- For $\alpha > 0$ and $0 \leq \beta < 1$, the hydrodynamic limit and the equilibrium fluctuations for $g(n) = \alpha n^{-\beta}$ should be driven by the heat equation periodic boundary conditions, achieving the same limits we have obtained for $g(n) = 1 + o(n)$.
- For $\alpha > 0$ and $\beta = 1$, the hydrodynamic limit and the equilibrium fluctuations for $g(n) = \alpha n^{-\beta}$ should be driven by (5.1) in the same sense of [6,8,7] with the correction of $1/2$ in the boundary condition.

6. Hydrodynamics of the SSEP with k neighboring slow bonds

Here we characterize the hydrodynamic behavior of the SSEP with k neighboring slow bonds, see Fig. 2 for an illustration of the rates in the case $k = 3$. This is an additional result we append in order to support the conjecture presented in Section 5. We point out that, for the regime $\beta = 1$, the result presented here is *not* a corollary of [6,8], since here we consider k *neighboring* slow bonds, while the mentioned references considered *macroscopically separated* slow bonds.

The notation and topology issues will be the same as those we have considered in this paper. Fix k a positive integer. The SSEP with k neighboring slow bonds is the Markov process on $\{0, 1\}^{\mathbb{T}_n}$ defined through the generator

$$\mathfrak{L}_n f(\eta) = \sum_{x=0}^{k-1} \frac{\alpha}{n^\beta} [f(\eta^{x, x+1}) - f(\eta)] + \sum_{\substack{x \in \mathbb{T}_n \\ x \notin \{0, \dots, k-1\}}} [f(\eta^{x, x+1}) - f(\eta)]$$

acting on functions $f : \{0, 1\}^{\mathbb{T}_n} \to \mathbb{R}$.

Definition 6. Let $\rho_0 : \mathbb{T} \to [0, 1]$ be a measurable function. We say that ρ is a weak solution of the heat equation with Robin's boundary conditions given by

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t \geq 0, \ u \in (0, 1), \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = \dfrac{\alpha}{k}(\rho(t, 0^+) - \rho(t, 0^-)), & t \geq 0, \\ \rho(0, u) = \gamma(u), & u \in (0, 1), \end{cases} \tag{6.1}$$

if ρ belongs to $L^2(0, T; \mathcal{H}^1)$ and for all $t \in [0, T]$ and for all $H \in C^2[0, 1]$, such that

$$\partial_u H(0^+) = \partial_u H(0^-) = \frac{\alpha}{k}(H(0^+) - H(0^-)), \tag{6.2}$$

holds that

$$\langle \rho_t, H \rangle - \langle \gamma, H \rangle - \int_0^t \langle \rho_s, \partial_u^2 H \rangle ds = 0.$$

Proposition 6.1. *For each $n \in \mathbb{N}$, let μ_n be a Bernoulli product measure on $\{0, 1\}^{\mathbb{T}_n}$ as in* (2.8). *Then, for any $t > 0$, for every $\delta > 0$ and every $H \in C(\mathbb{T})$, it holds that*

$$\lim_{n \to \infty} \mathbb{P}_{\mu_n}\left\{\eta. : \left|\frac{1}{n}\sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \eta_t(x) - \int_{\mathbb{T}} H(u) \rho(t, u)du\right| > \delta\right\} = 0, \tag{6.3}$$

where

- *if $0 \leq \beta < 1$, the function ρ is the unique weak solution of* (2.4);
- *if $\beta = 1$, the function ρ is the unique weak solution of* (6.1);
- *if $\beta > 1$, the function ρ is the unique weak solution of* (2.6).

Proof. As usual, the proof consists in proving tightness of the process induced by the empirical measure, plus the uniqueness of the limit points.

Tightness can be handled in same way as we have done in Section 3.1. Characterization of the limit points for the cases $\beta \in [0, 1)$ and $\beta \in (1, \infty)$ follows closely the steps of [6,8]. The case $\beta = 1$ is tricky and described in more detail below.

Let $G_n : \{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \to \mathbb{R}$ be some function depending on n. Performing elementary computations, $n^2 \mathfrak{L}_n \langle \pi_s^n, G_n \rangle$ can be rewritten as the sum of

$$\left[n\left(G_n\left(\frac{k+1}{n}\right) - G_n\left(\frac{k}{n}\right)\right) + \alpha n^{1-\beta}\left(G_n\left(\frac{k-1}{n}\right) - G_n\left(\frac{k}{n}\right)\right)\right]\eta_s(k)$$

$$+ \sum_{j=1}^{k-1}\left[\alpha n^{1-\beta}\left(G_n\left(\frac{j+1}{n}\right) - G_n\left(\frac{j}{n}\right)\right)\right.$$

$$+ \alpha n^{1-\beta}\left(G_n\left(\frac{j-1}{n}\right) - G_n\left(\frac{j}{n}\right)\right)\Big]\eta_s(j)$$

$$+ \left[\alpha n^{1-\beta}\left(G_n\left(\frac{1}{n}\right) - G_n\left(\frac{0}{n}\right)\right) + n\left(G_n\left(\frac{-1}{n}\right) - G_n\left(\frac{0}{n}\right)\right)\right]\eta_s(0) \tag{6.4}$$

and

$$n \sum_{\substack{x \in \mathbb{T}_n \\ x \notin \{0,\ldots,k\}}}\left(G_n\left(\frac{x+1}{n}\right) + G_n\left(\frac{x-1}{n}\right) - 2G_n\left(\frac{x}{n}\right)\right)\eta_s(x). \tag{6.5}$$

Let $H \in C^2[0, 1]$ satisfying (6.2). We define G_n by

$$G_n\left(\frac{x}{n}\right) = \begin{cases} H\left(\frac{x}{n}\right), & \text{if } x \in \{k+1, \ldots, n-1\}, \\ H(0^-) + \frac{x}{k}(H(0^+) - H(0^-)), & \text{if } x \in \{0, \ldots, k\}. \end{cases} \tag{6.6}$$

In other words, the function G_n is equal to H outside the region where the slow bonds are contained. At the sites $\{0, 1, \ldots, k\}$, the function G_n is a linear interpolation of $H(0^+)$ and $H(0^-)$.

Since $H \in C^2[0, 1]$, (6.5) is close to $\langle \pi_s^n, \partial_u^2 H \rangle$. We claim now that (6.4) converges to zero, as $n \to \infty$. First, notice that (6.6) tells us that

$$\sum_{j=1}^{k-1} \left[\alpha n^{1-\beta} \left(G_n \left(\frac{j+1}{n} \right) - G_n \left(\frac{j}{n} \right) \right) + \alpha n^{1-\beta} \left(G_n \left(\frac{j-1}{n} \right) - G_n \left(\frac{j}{n} \right) \right) \right] \eta_s(j)$$

is null. Let us analyze the remaining terms in (6.4). Since $\beta = 1$, the term

$$n \left(G_n \left(\frac{k+1}{n} \right) - G_n \left(\frac{k}{n} \right) \right) + \alpha n^{1-\beta} \left(G_n \left(\frac{k-1}{n} \right) - G_n \left(\frac{k}{n} \right) \right)$$

converges to

$$\partial_u H(0^+) + \frac{\alpha}{k} \left(H(0^-) - H(0^+) \right),$$

which vanishes by (6.2). The same analysis assures that

$$\alpha n^{1-\beta} \left(G_n \left(\frac{1}{n} \right) - G_n \left(\frac{0}{n} \right) \right) + n \left(G_n \left(\frac{-1}{n} \right) - G_n \left(\frac{0}{n} \right) \right)$$

converges to zero as $n \to \infty$. Provided by this claim and similar arguments of those in Section 5 one can conclude the proof. $\square$

Acknowledgments

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