Universidade Federal de Minas Gerais - UFMG
Instituto de Ciências Exatas - ICEX
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Tese de Doutorado

Large Deviations for a Reaction-Diffusion System in the Supremum Norm

Luana Amaral Gurgel

Orientador: Prof. Bernardo Nunes Borges de Lima.
Orientador: Prof. Tertuliano Franco Santos Franco.

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Aos meus pais
e meu marido Túlio.

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"Se um dia tiver que escolher entre o mundo e o amor lembre-se: se escolher o mundo ficará sem o amor, mas se escolher o amor com ele você conquistará o mundo."

(Albert Einstein)

Resumo

Apresentamos aqui um princípio de grandes desvios com a norma do supremo para o limite de altas densidades de um sistema de passeios aleatórios sobrepostos e independentes com uma dinâmica de nascimento e morte, que exibe como função taxa uma versão semi-linearizada da função taxa de [11], que lidava com grandes desvios dos processos de exclusão sobrepostos com uma dinâmica de nascimento e morte. Devido à força natural da topologia do conjunto, a prova do limite inferior se torna muito simples em comparação com as provas padrões do limite inferior do ponto de vista hidrodinâmico (como em [12], [Capítulo 10], por exemplo), além do próprio resultado, que pode ter aplicações devido à ampla ocorrência de equações diferenciais parciais de reação-difusão. A principal novidade do presente trabalho consiste em fornecer uma estratégia para estender a abordagem original de alta densidade (como em [1, 3, 4, 8, 13, 14], por exemplo), originalmente desenvolvido para sistemas de difusão simétrica, para sistemas fracamente assimétricos.

Palavras-chave: Reação-difusão, grandes desvios, dinâmica de nascimento-morte.

Abstract

We present here a full large deviations principle in the supremum norm for the high density limit of a system of independent random walks superposed with a birth-and-death dynamics, which exhibits as a rate function a semi-linearised version of the rate function of [11], which dealt with large deviations of exclusion processes superposed with birth-and-death dynamics. Due to the strong nature of the topological setting, the proof of the lower bound turns to be very simple in comparison with the standard proofs of lower bound in the hydrodynamic point of view (as in [12, Chapter 10], for instance), aside of the result itself, which may have many applications due the broad occurrence of reaction-diffusion partial differential equations. The main novelty of the present work consists in providing a strategy to extend the original high density approach (as in [1, 3, 4, 8, 13, 14] for instance), originally developed to systems of symmetric diffusion, to weakly asymmetric systems.

Keywords: Reaction-diffusion, large deviations, birth-and-death dynamics.

Contents

Chapter 1

Introduction

Since the early works of Dobrushin (see [5]) and the seminal paper of Guo, Papanicolau and Varadhan (see [9]), an entire theory on scaling limits of interacting particle systems has been established, see the reference book [12]. Such a subject has its great importance in the context of statistical mechanics, in understanding the behaviour of macroscopic systems by means of its microscopic interactions, but has also many connections with partial differential equations, probability theory and even combinatorics (see [16]).

At same epoch the hydrodynamic limit (see [12] on the subject) started to be developed, some works were published in a close topic sometimes called *high density limit*, also in the context of scaling limit of interacting particle systems, as [1, 3, 4, 13, 14] for instance. The main difference between the hydrodynamic limit and the high density limit can be resumed as follows: while in hydrodynamic limit space and time are rescaled in order to obtain a macroscopic limit, in the high density limit, space, time *and the initial quantity of particles per site* are rescaled, see the survey [7] for a discussion about. Of course, each context requires a different topology setting. Whilst the hydrodynamic limit usually deals with convergence of measures, Schwartz distributions and Sobolev norms, the high density limit deals usually with Sobolev norms, but also allows the supremum norm set-up, see [4].

In opposition to the hydrodynamic limit, which has been continuously studied since its beginning, the high density limit felt in disuse for many years. Its was probably due to the following reason: the powerful *Varadhan's Entropy Method* allowed the study of systems of non-linear diffusion[1], while the high density limit approach was restricted to systems of linear diffusion. Basically, independent random walks with some superposed dynamics, as the birth-and-death dynamics, for example. Actually, the high density approach is heavily based on the smoothing properties of the discrete heat kernel, which explains this restriction to independent random walks.

On the other hand, despite its symmetric nature, the high density limit offers some particular perspectives, which would be difficult to be followed in the hydrodynamic setting. For example, in [8], it was considered a system exhibiting explosion in finite time. Since the hydrodynamic limit techniques are mainly based on

[1]As well as some other methods, as the *Yau's Relative Entropy Method*, see [12].

averages, the system of [8] would thea hard topic to be analysed in the hydrodynamic point of view since there is no finite expectation of standard observables. In the intersection, some recent works also rescale the initial quantity of particles per site, which may be interpreted as a kind of high density limit, as [10] for example.

The main result we present here is a large deviations principle for the law of large numbers of [4], which consists in the high density limit in the supremum norm for a system of independent random walks on the discrete torus superposed with a birth and death dynamics. Actually, following some observations of [8], weaking some assumptions on the birth and and death rates, we consider a slightly more general system than that in [4], but we may say that the model we consider is essentially that one of [4]. As usual in large deviations, an important ingredient of the proof is a law of large numbers for a class of perturbations of the original model, which is an interesting result by itself. Since the high density limit was originally designated for *systems of symmetric diffusion* (independent random walks superposed with some extra dynamics), we can say that the more challenging step in our proof is to reach the law of large numbers for the perturbed processes, which are *weakly asymmetric systems*. Following some remarks from [8] we were also able to assure that the law of large numbers for the perturbed processes takes place in the almost sure sense, which is an important feature.

The rate function we obtain in the large deviations is a spatially linearised version of the rate function of [11], which dealt with large deviations of a superposition of Glauber and Kawasaki dynamics. However, this resemblance is limited to this observation: since [11] works on the hydrodynamic limit and we work on the high density limit, the technical challenges we face here are very distinct of those in [11].

Due to the strong topological nature of the supremum norm and the obtained almost sure convergence, some usual difficulties when proving large deviations for the hydrodynamic point of view do not appear in this setting, considerably simplifying the upper and lower bound arguments, except when achieving the *exponential tightness*, which demanded some extra effort. For example, no superexponential replacement lemmas are required here. On the other hand, as aforementioned, the convergence of the perturbed processes, which is in general a standard procedure in the hydrodynamic limit (for the exclusion process for instance, see [12, Chapter 10]), here is an obstacle to be overcome.

Apart of the result itself, which is relevant due the broad occurrence of reaction-diffusion partial differential equations and the importance of the supremum norm for simulations, the main novel of the present work consists in providing a strategy to extend the original high density approach (as in [1, 3, 4, 8, 13, 14]), originally developed to systems of symmetric diffusion, to spatially weakly asymmetric systems. Before explaining our strategy for weakly asymmetric systems, let us hand-waving resume the way in [4] of proving the high density limit.

The first ingredient is to show that the solution of a spatially discrete version of the limit PDE is close to the solution of the limiting PDE. After that, we must study the martingales associated to the projection at each site. Due to the scale setting of parameters, in opposition to the Entropy Method, showing that the quadratic variation of those martingales vanish does not suffice to lead to the convergence in the supremum norm. From these martingales and the presence of the discrete Laplacian, we obtain integral equations via the Duhamel's Principle, which involve the heat semigroup instead of the Laplacian operator. Then, by providing some suitable estimates on the random term of these equations and recalling smoothing properties of the heat semi-group allows to get the desired convergence in the supremum norm.

For our work we use this same process to get the high density limit for the weakly asymmetric systems, however, as has been said, the asymmetry in the system causes some difficulties. Having the high density limit for a class of perturbed processes we proceed with the large deviations principle. Before we need to find the expression for the Radon-Nikodym derivative between the original process and the perturbed process. Knowing the existence of the Radon-Nikodyn derivative we can prove the large deviations upper bound, here arises the need to show that the sequence of measures of process is exponentially tight. For the lower bound, we separated in two cases. First, we consider that profiles, which are a solution of the differential equation considering the perturbed process, are smooth functions. Finally, we will consider more general profiles but include additional assumptions on the process birth and death rates and about the parameter that indicates the initial mean number of particles.

This work is divided into three parts. In the Chapter 2, we will present the models of interacting particle systems with reaction and diffusion used here and enunciate out the main results of this work. In the Chapter 3, we will do all the necessary steps to prove the high desity limit for the perturbed process and finally, in the Chapter 4, we find the Radon-Nikodym derivative and we demonstrate the large deviations principle.

Chapter 2

Models and Results

In this chapter we will present the model of interacting particle systems with reaction and diffusion and the result made in [4, 8] that motivated this work. Next, we will present the model for the perturbed process which will be a modification of the original model and the high density limit for perturbed process. Finally, we will state the principle of large deviations obtained for this model.

2.1 Introduction

We fix some notation that will be used at various moments throughout the text. Consider by $g = O(f)$ we mean that the function g is bounded in modulus by a constant times the function f, where the constant may change from line to line, or even may represent different functions in the same formula, but it will never depend on the parameter of interest. The spatial first and second derivates on space will be denoted by ∇ and Δ. However, sometimes we will write also ∂_x and ∂_x^2 instead of ∇ and Δ to better differentiate it of discrete derivatives to be later defined.

Denote by $\mathbb{T}_N = \mathbb{Z}/(N\mathbb{Z})$ the discrete torus with N sites and by $\mathbb{T}$ denote the continuous torus $\mathbb{R}/\mathbb{Z} = [0, 1)$, where the point 0 is identified with the point 1. Let $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ be two Lipschitz functions such that $d(0) = 0$. Throughout this paper, $\ell = \ell(N)$ will be a parameter meaning the initial average number of particles at any given site of $\mathbb{T}_N$. We define below the continuous-time Markov chain $\big(\eta(t)\big)_{t\geq 0}$ with state space $\Omega_N = \mathbb{N}^{\mathbb{T}_N}$ by

$$\big(\eta(t)\big)_{t\geq 0} \;=\; \big(\eta_1(t),\, \dots\, ,\, \eta_N(t)\big)_{t\geq 0}\,,$$

where $\eta_k(t)$ means the quantity of particles at the site k at the time t. The jump rates of the process will be given by:

- a particle jumps from k for $k+1$ at rate $N^2\eta_k$, as in Figure 2.1 ,
- a particle jumps from k for $k-1$ at rate $N^2\eta_k$, as in Figure 2.2,
- a new particle is created at site k at rate $\ell b(\ell^{-1}\eta_k)$, as in Figure 2.3,
- a particle is destroyed at site k at rate $\ell d(\ell^{-1}\eta_k)$, if $\eta_k \geq 1$, as in Figure 2.4.

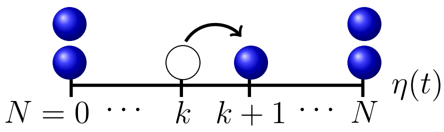

Figure 2.1: Particle jump to the right.

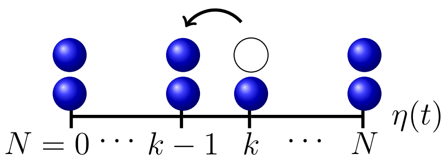

Figure 2.2: Particle jump to the left.

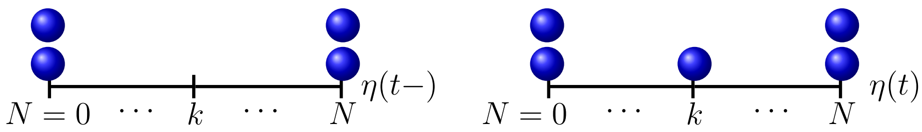

Figure 2.3: Particle is created.

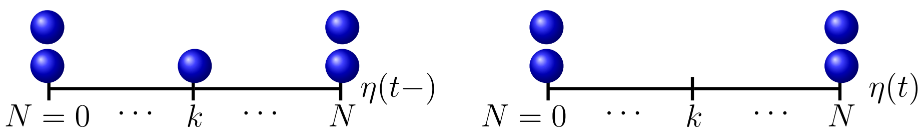

Figure 2.4: Particle is destroyed.

Throughout the entire thesis, it is fixed a time-horizon $T > 0$. Let $\mathcal{D}\big([0,T],\Omega_N\big)$ be the path space of càdlàg time trajectories taking values on Ω_N. For short, we will denote this space just by $\mathcal{D}_{\Omega_N}$. Given a measure μ_N on Ω_N, denote by $\mathbb{P}_N$ the probability measure on $\mathcal{D}_{\Omega_N}$ induced by the initial state μ_N and the Markov process $\{\eta(t) : t \geq 0\}$. Expectation with respect to $\mathbb{P}_N$ will be denoted by $\mathbb{E}_N$. The object we are interested in this thesis is the spatial density X^N of particles, defined as follows. For $k \in \mathbb{T}_N$, denote $x_k = k/N$. Let

$$X^N(t,x_k) \;=\; \ell^{-1}\eta_k(t) \tag{2.1}$$

and, for $x_k < x < x_{k+1}$, we define $X^N(t,x)$ by means of a linear interpolation, i.e.

$$X^N(t,x) \;=\; (Nx-k)X^N(t,x_{k+1}) + (k+1-Nx)X^N(t,x_k)\,. \tag{2.2}$$

In [4, 8] it was essentially proved the following law of large numbers for the density of particles.

Theorem 2.1.1 ([4, 8]). *Let $u(t,x)$ be the strong solution of the following PDE:*

$$\begin{cases} \partial_t u \;=\; \Delta u + f(u) & (t,x) \in [0,T]\times\mathbb{T}\,, \\ u(0,x) \;=\; \gamma(x) \geq 0 & x \in \mathbb{T}\,, \end{cases} \tag{2.3}$$

Let $b,d : \mathbb{R}_+ \to \mathbb{R}_+$ be Lipschitz C^1 functions with $d(0) = 0$ and $f = b - d$, and let $\gamma : \mathbb{T} \to \mathbb{R}_+$ be a C^4 profile. Assume that

(1) $\|X^N(0,\cdot)-\gamma(\cdot)\|_\infty \to 0$ *almost surely;*

(2) *for any* $c>0$, $\ell=\ell(N)$ *satisfies* $\sum_{N\geq 0} N^3 e^{-c\ell}<\infty$.

Then, for any $T>0$,

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\|X^N(t,\cdot)-u(t,\cdot)\|_\infty \;=\; 0 \quad \textit{almost surely.} \tag{2.4}$$

The assumption **(1)** above allows us to interpret the parameter ℓ as the initial order of particles per site, from where comes the terminology *high density limit* (see [14] for instance). In contrast with the hydrodynamic limit (see [12]), where only time and space are rescaled, here *time, space and the initial quantity of particles per site are rescaled*, which permits convergence in the supremum norm.

In [4], the above result was proved under the assumption that b, d were polynomial functions and the leader coefficient of $f=b-d$ is negative, being the convergence in probability. In [8], this result was proved the setting where b and d were smooth but not necessarily bounded, with almost sure convergence.

The specific statement given above is a particular case of our Theorem 2.2.1 which we enunciate ahead. The subtle differences in the hypothesis will be relevant when dealing with large deviations.

2.2 High density limit for a class of perturbed processes

In the proof of large deviations, a law of large numbers for a class of perturbations of the original process is naturally required, which is an interesting result by itself. For the reaction-diffusion model we study here, inspired by the process of [11] the perturbed process will be the following. Given $H\in C^{1,2}$, we define the continuous-time Markov chain $\big(\eta(t)\big)_{t\geq 0}$ with state space $\Omega_N=\mathbb{N}^{\mathbb{T}_N}$ by

$$\big(\eta(t)\big)_{t\geq 0} \;=\; \big(\eta_1(t),\,\dots\,,\,\eta_N(t)\big)_{t\geq 0}\,,$$

where $\eta_k(t)$ means the quantity of particles at site k at time t as before, and the jump rates of the process are given by:

- a particle jumps from k for $k+1$ with rate $N^2\eta_k \exp\Big\{H\big(t,\frac{k+1}{N}\big)-H\big(t,\frac{k}{N}\big)\Big\}$,
- a particle jumps from k for $k-1$ with rate $N^2\eta_k \exp\Big\{H\big(t,\frac{k-1}{N}\big)-H\big(t,\frac{k}{N}\big)\Big\}$,
- a new particle is created at site k with rate $\ell b(\ell^{-1}\eta_k)\exp\Big\{H\big(t,\frac{k}{N}\big)\Big\}$,
- a particle is destroyed at site k with rate $\ell d(\ell^{-1}\eta_k)\exp\Big\{-H\big(t,\frac{k}{N}\big)\Big\}$, if $\eta_k\geq 1$.

Note that this Markov chain actually depends on H. However, to not overload notation, this dependence will be dropped. Given a measure μ_N on Ω_N, denote by $\mathbb{P}_N^H$ the probability measure on $\mathcal{D}_{\Omega_N}$ induced by the initial state μ_N and the Markov process $\{\eta(t): t\geq 0\}$ above. Expectation with respect to $\mathbb{P}_N^H$ will be denoted by $\mathbb{E}_N^H$.

Let $\psi : [0,T] \times \mathbb{T} \to \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} \partial_t \psi \;=\; \partial_x^2 \psi - 2\partial_x\big(\psi\, \partial_x H\big) + e^{H} b(\psi) - e^{-H} d(\psi)\,, & (t,x) \in [0,T] \times \mathbb{T}\,, \\ \psi(0,x) \;=\; \gamma(x), & x \in \mathbb{T}. \end{cases} \tag{2.5}$$

Assuming that $H \in C^{1,2}$, $b, d \in C^1$ and γ is Holder continuous in $\mathbb{T}$, there exists a unique solution of the PDE (2.5), which we denote by ψ, see [15, Chapter II, Section 2.3]. We point out that the PDE above can be understood as a linearised version of the PDE in [11, (2.11)].

Next, we state the high density limit for the perturbed process. As before, $X^N(t) = X^N(t,x)$ is equal to $\eta_k(t)/\ell$ for $x = k/N$ and linearly interpolated otherwise. Of course, this process depends on H, whose dependence is omitted. For the High density limit for perturbed processes, we consider $X^N(0,\cdot)$ are random profiles for each $N \in \mathbb{N}$. Let μ_N the sequence measure that describes this initial processes.

Theorem 2.2.1 (High density limit for perturbed processes)**.** *Let $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ be Lipschitz C^1 functions with $d(0) = 0$, let $H \in C^{1,2}$ and let $\gamma : \mathbb{T} \to \mathbb{R}_+$ be a C^4 profile. Assume the following conditions:*

(A1) *The sequence of initial measures μ_N is such that*

$$\|X^N(0,\cdot) - \gamma(\cdot)\|_\infty \to 0\,, \qquad \textit{almost surely as } N \to \infty\,. \tag{2.6}$$

(A2) *The parameter $\ell = \ell(N)$ satisfies*

$$\frac{N^{\|\partial_x H\|_\infty^2/\pi^2} \log N}{\ell} \to 0\,, \qquad \textit{as } N \to \infty\,. \tag{2.7}$$

Then,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|X^N(t,\cdot) - \psi(t,\cdot)\|_\infty \;=\; 0\,, \qquad \textit{almost surely as } N \to \infty\,,$$

where ψ is the strong solution of (2.5)*.*

Remark 2.2.2. Note that there is no further hypothesis on the sequence of initial measures μ_N aside of (2.6). As an example of a sequence of initial measures, one may consider μ_N as a product measure of Poisson distributions whose parameter at the site $x \in \mathbb{T}$ is given by $\ell\gamma(x/N)$. However, as we are interested in dynamical large deviations, throughout the thesis we will assume that μ_N is a deterministic sequence, that is, each μ_N is a delta of Dirac on some configuration.

Remark 2.2.3. Let us discuss the meaning of **(A2)**. Taking $\ell(N) = N^\alpha$ with $\alpha > 0$, condition (2.7) holds once $\|\partial_x H\|_\infty < \pi\sqrt{\alpha}$. This may look weird at a first glance, but it is not completely unexpected. The role of H is to introduce an asymmetry in the system. Since the density limit approach is heavily founded on the smoothing properties of the discrete heat kernel (which is associated to the *symmetric* random walk), it is somewhat reasonable to have a competition between the speed of $\ell(N)$ and the strength of the function H.

On the other hand, under the hypothesis $\ell = \ell(N) \geq e^{cN}$ for some constant c, the high density limit holds for any perturbation $H \in C^{1,2}$.

2.3 Large deviations results

In the present thesis we prove the large deviations principle for the law of large numbers of Theorem 2.1.1, which we state in the sequel. Denote by $C(\mathbb{T})$ the Banach space of continuous functions $H : \mathbb{T} \to \mathbb{R}$ under the supremum norm $\| \cdot \|_\infty$. Denote by $C^{1,2} \stackrel{\text{def}}{=} C^{1,2}\big([0,T] \times \mathbb{T}\big)$ the set of functions $H : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that H is C^2 in space and C^1 in time. Let $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ be the Skorohod space of càdlàg trajectories taking values on $C(\mathbb{T})$. We define now the functional $J_H : \mathcal{D} \to \mathbb{R}$ by

$$
\begin{aligned}
J_H(u) \;=\; & \int_{\mathbb{T}} \Big[H(t,x)u(t,x) - H(0,x)u(0,x) \Big]\, dx \\
& + \int_0^t \int_{\mathbb{T}} \Big[-u(s,x)\Big(\partial_s H(s,x) + \Delta H(s,x) + \big(\nabla H(s,x)\big)^2\Big) \\
& \qquad + b\big(u(s,x)\big)\big(1 - e^{H(s,x)}\big) + d\big(u(s,x)\big)\big(1 - e^{-H(s,x)}\big)\Big]\, dx\, ds
\end{aligned}
\tag{2.8}
$$

Recalling that $\gamma : \mathbb{T} \to \mathbb{R}_+$ is the non negative C^4 function which appears in in the Theorems 2.1.1 and 2.2.1, let $\boldsymbol{I} : \mathcal{D} \to [0, +\infty]$, rate function which depends on γ, be given by

$$
\boldsymbol{I}(u) \;=\; \begin{cases} \sup\limits_{H \in C^{1,2}} J_H(u)\,, & \text{if } u(0,\cdot) = \gamma(\cdot), \\ +\infty\,, & \text{otherwise.} \end{cases}
$$

Initially we will prove the principle of large deviations for a subspace of $\mathcal{D}\big([0,T], C(\mathbb{T})\big)$, denoted by $\mathcal{D}^\alpha_{\text{pert}}$, which we define below.

Definition 2.3.1. *Denote by* $\mathcal{D}^\alpha_{\text{pert}} \subseteq \mathcal{D} = \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ *the set of all profiles* $\psi : [0,T] \times \mathbb{T} \to \mathbb{R}$ *satisfying:*

- $\psi \in C^{2,3}$,
- $\psi \geq \varepsilon$ *for some* $\varepsilon > 0$,
- *there exists a function* $H \in C^{1,2}$, *with* $\|\partial_x H\|_\infty \leq \pi\sqrt{\alpha}$ *such that* ψ *is the solution of* (2.5).

We are in position to state the main result of this thesis. Let P_N be the probability measure on the $\mathcal{D}$ induced by the stochastic process $X^N(t)$ defined by (2.1) and (2.2).

Theorem 2.3.2 (Large deviations principle). *Assume the hypothesis of Theorem 2.1.1 and additionally assume that* $X^N(0,\cdot)$ *are deterministic profiles for each* $N \in \mathbb{N}$. *Let* $\ell = \ell(N) = N^\alpha$ *for some fixed* $\alpha > 0$. *Then,*

1) For every closed set $\mathcal{C} \subseteq \mathcal{D}$,

$$
\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N(\mathcal{C}) \;\leq\; -\inf_{u \in \mathcal{C}} \boldsymbol{I}(u)\,. \tag{2.9}
$$

2) For every open set $\mathcal{O} \subseteq \mathcal{D}$,

$$\liminf_{N\to\infty} \frac{1}{\ell N} \log P_N(\mathcal{O}) \;\geq\; -\inf_{u\in\mathcal{O}\cap\mathcal{D}^{\alpha}_{\text{pert}}} \boldsymbol{I}(u)\,. \tag{2.10}$$

We note that the assumption that the initial conditions are deterministic prevents the occurrence of large deviations from the initial profile, also known as *statical large deviations*. Our main interest here are the *dynamical large deviations*, that is, the large deviations coming from the dynamics. Moreover, in the previous result, the clearly more relevant result is the upper bound. The lower bound holds only over sets intersected with $\mathcal{D}^{\alpha}_{\text{pert}}$, which has no explicit representation.

In the case $\ell = \ell(N)$ grows at least exponentially, we were able to provide a full large deviations principle.

Theorem 2.3.3. *Assume the hypothesis of Theorem 2.1.1 and additionally assume that $X^N(0,\cdot)$ are deterministic profiles for each $N \in \mathbb{N}$ and b and d are concave functions. Let $\ell = \ell(N) \geq e^{cN}$ for some constant $c > 0$. Then,*

1) For every closed set $\mathcal{C} \subseteq \mathcal{D}$,

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N(\mathcal{C}) \;\leq\; -\inf_{u\in\mathcal{C}} \boldsymbol{I}(u)\,. \tag{2.11}$$

2) For every open set $\mathcal{O} \subseteq \mathcal{D}$,

$$\liminf_{N\to\infty} \frac{1}{\ell N} \log P_N(\mathcal{O}) \;\geq\; -\inf_{u\in\mathcal{O}} \boldsymbol{I}(u)\,. \tag{2.12}$$

Chapter 3

High density limit for the perturbed process

In this chapter we will introduce the ingredients to prove Theorem 2.2.1. First, we will do the semidiscrete approximation ψ^N of the PDE (2.5), which will be directly used to calculate the difference between the solution of PDE (2.5) and the spatial density of particles X^N. After that, through the infinitesimal generator of process, we calculate the Dynkin martingale of the (perturbed) Markov process present in Section 2.2. For this martingal and for ψ^N, we prove a version of Duhamel's principle and finally, having all these tools, prove Theorem 2.2.1 in last section.

3.1 Semi-discrete scheme

In this section we consider a spatial discretization of the PDE (2.5), keeping continuous the time variable. We follow the ideas used in [8]. Our goal is to prove convergence of such spatial discretization to the solution of the partial differential equation (2.5), which is an ingredient in the proof of the high density limit for the perturbed process. In fact, the proof of Theorem 2.2.1 is done in two steps. First, we prove that the solution of the PDE (2.5) is close to the solution of its spatial discretization; then, we will prove that the (deterministic) solution of the spatial discretization is close to the random density of particles defined by $X^N(t)$.

We denote $H_k = H_k(t) = H(k/N, t)$, $\psi_k = \psi(x_k, t)$ and by $S^N_{\pm 1}$ the shifts of $\pm N^{-1}$. That is,

$$S^N_1 f\big(s, \tfrac{k}{N}\big) \;=\; f\big(s, \tfrac{k+1}{N}\big) \qquad \text{and} \qquad S^N_{-1} f\big(s, \tfrac{k}{N}\big) \;=\; f\big(s, \tfrac{k-1}{N}\big)\,.$$

We define the semidiscrete approximation $\psi^N(t) = \big(\psi^N_1(t), \dots, \psi^N_N(t)\big)$ of the PDE (2.5) as the solution of the following system of ODE's:

$$\begin{cases} \partial_t \psi^N_k \;=\; N^2\big(\psi^N_{k+1} - 2\psi^N_k + \psi^N_{k-1}\big) - N\big(\psi^N_{k+1} - \psi^N_{k-1}\big)\partial_x H_k \\ \qquad\qquad -\frac{1}{2}\Big(S^N_1 + S^N_{-1} + 2\Big)\psi^N_k \partial^2_x H_k + e^{H_k} b\big(\psi^N_k\big) - e^{-H_k} d\big(\psi^N_k\big)\,, \quad k \in \mathbb{T}_N\,, \\ \psi^N_k(0) \;=\; \gamma(k/N)\,, \quad k \in \mathbb{T}_N\,. \end{cases} \tag{3.1}$$

Proposition 3.1.1. *Let ψ be the solution of* (2.5) *and let ψ^N be the solution of the semidiscrete approximation* (3.1)*. Then, there exists a positive constant C such that, for every N large enough,*

$$\sup_{t\in[0,T]}\max_{k\in\mathbb{T}_N}\left|\psi_k^N(t)-\psi_k(t)\right| \;\leq\; CN^{-1}\,. \tag{3.2}$$

To prove the result above we will need an auxiliary lemma about solutions for the following system of ODE's, all of them considered in the time interval $[0,T]$,

$$\begin{cases}\partial_t\varphi_k \;=\; N^2\big(\varphi_{k+1}-2\varphi_k+\varphi_{k-1}\big)-N\big(\varphi_{k+1}-\varphi_{k-1}\big)\partial_x H_k\\ \qquad\quad +C_*\big(|\varphi_k|+|\varphi_{k+1}|+|\varphi_{k-1}|+N^{-1}\big)\,,\\ \varphi_k(0) \;=\; 0,\;\; k\in\mathbb{T}_N\,.\end{cases} \tag{3.3}$$

We say that $\overline{\varphi}=(\overline{\varphi}_1,\cdots,\overline{\varphi}_n)$ is *supersolution* of (3.3) if

$$\begin{cases}\partial_t\overline{\varphi}_k \geq N^2\big(\overline{\varphi}_{k+1}-2\overline{\varphi}_k+\overline{\varphi}_{k-1}\big)-N\big(\overline{\varphi}_{k+1}-\overline{\varphi}_{k-1}\big)\partial_x H_k\\ \qquad\quad +C_*\big(|\overline{\varphi}_k|+|\overline{\varphi}_{k+1}|+|\overline{\varphi}_{k-1}|+N^{-1}\big)\,,\\ \overline{\varphi}_k(0)\geq 0,\;\; k\in\mathbb{T}_N\,,\end{cases} \tag{3.4}$$

and we say that $\underline{\varphi}=(\underline{\varphi}_1,\cdots,\underline{\varphi}_n)$ is *subsolution* of (3.3) if

$$\begin{cases}\partial_t\underline{\varphi}_k \;\leq\; N^2\big(\underline{\varphi}_{k+1}-2\underline{\varphi}_k+\underline{\varphi}_{k-1}\big)-N\big(\underline{\varphi}_{k+1}-\underline{\varphi}_{k-1}\big)\partial_x H_k\\ \qquad\quad +C_*\big(|\underline{\varphi}_k|+|\underline{\varphi}_{k+1}|+|\underline{\varphi}_{k-1}|+N^{-1}\big)\,,\\ \underline{\varphi}_k(0) \;\leq\; 0,\;\; k\in\mathbb{T}_N\,.\end{cases} \tag{3.5}$$

Above,

$$C_* \;\stackrel{\text{def}}{=}\; \max\Big\{\|e^H\|_\infty\cdot\|b\|_L,\|e^{-H}\|_\infty\cdot\|d\|_L,\|\partial_x H\|_\infty,\|\partial_x^2 H\|_\infty,\|\partial_x\psi\|_\infty\Big\}\,, \tag{3.6}$$

where ψ is the solution of PDE (2.5) and $\|b\|_L$ and $\|d\|_L$ are the Lipschitz constants of b and d, respectively. The necessity of these notions and the definition (3.6) will be made clear later in the proof of Proposition 3.1.1.

Lemma 3.1.2 (Principle of sub and supersolutions)**.** *Let $\overline{\varphi}$, $\underline{\varphi}$, φ be a supersolution, a subsolution and a solution of* (3.3)*, respectively. Then there exists $N_0\in\mathbb{N}$ such that , for any $N\geq N_0$,*

$$\overline{\varphi}_k(t) \;\geq\; \varphi_k(t) \;\geq\; \underline{\varphi}_k(t) \tag{3.7}$$

for any $k\in\mathbb{T}_N$ and any $t\in[0,T]$.

Proof. We will prove only that $\overline{\varphi}\geq\varphi$, being the second inequality analogous. We claim that it is enough to prove $\overline{\varphi}\geq\varphi$ assuming strict inequalities in (3.4) and in (3.7). In fact, assume that $\overline{\varphi}$ is a supersolution, that is, it satisfies (3.4) and define $\widetilde{\varphi}(t)=\overline{\varphi}(t)+\varepsilon t$. Hence,

$$
\begin{aligned}
\partial_t \widetilde{\varphi} \;=\; \partial_t \overline{\varphi} + \varepsilon \;&\geq N^2(\overline{\varphi}_{k+1} - 2\overline{\varphi}_k + \overline{\varphi}_{k-1}) - N(\overline{\varphi}_{k+1} - \overline{\varphi}_{k-1})\partial_x H_k \\
&\quad + C_*\big(|\overline{\varphi}_k| + |\overline{\varphi}_{k+1}| + |\overline{\varphi}_{k-1}| + N^{-1}\big) + \varepsilon \\
&\geq N^2(\widetilde{\varphi}_{k+1} - 2\widetilde{\varphi}_k + \widetilde{\varphi}_{k-1}) - N(\widetilde{\varphi}_{k+1} - \widetilde{\varphi}_{k-1})\partial_x H_k \\
&\quad + C_*\big(|\widetilde{\varphi}_k| + |\widetilde{\varphi}_{k+1}| + |\widetilde{\varphi}_{k-1}| + N^{-1}\big) - 3C_* t\varepsilon + \varepsilon\,.
\end{aligned}
$$

Therefore, $\widetilde{\varphi}$ is a (strictly) supersolution once $-3C_*t\varepsilon + \varepsilon > 0$ or, equivalently, if $t < 1/(3C_*)$. Partitioning the time interval $[0,T]$ into a finite number of intervals of length strictly smaller than $1/(3C_*)$ allows us to conclude that $\widetilde{\varphi}$ is a stricly supersolution in the time interval $[0,T]$. Hence $\widetilde{\varphi} > \varphi$ by assumption and letting $\varepsilon \searrow 0$ yields $\overline{\varphi} \geq \varphi$. This concludes the proof of the claim.

Suppose by contradiction that there is a first time t_* and a site $k \in \mathbb{T}_N$ such that for any $t < t_*$, $\overline{\varphi}_k(t) > \varphi_k(t)$ and $\overline{\varphi}_j(t) > \varphi_j(t)$ for any $j \neq k$, and $\overline{\varphi}_k(t_*) = \varphi_k(t_*)$. In this situation,

$$
\begin{aligned}
0 \;&\geq\; \partial_t \overline{\varphi}_k(t_*) - \partial_t \varphi_k(t_*) \\
&>\; N^2\Big(\overline{\varphi}_{k+1}(t_*) - \varphi_{k+1}(t_*) + \overline{\varphi}_{k-1}(t_*) - \varphi_{k-1}(t_*)\Big) \\
&\quad - N\Big(\overline{\varphi}_{k+1}(t_*) - \varphi_{k+1}(t_*) - \overline{\varphi}_{k-1}(t_*) + \varphi_{k-1}(t_*)\Big)\partial_x H_k \\
&\quad + C_*\Big(|\overline{\varphi}_{k+1}| - |\varphi_{k+1}| + |\overline{\varphi}_{k-1}| - |\varphi_{k-1}|\Big) \\
&\geq\; (N^2 - N\partial_x H_k)\Big(\overline{\varphi}_{k+1}(t_*) - \varphi_{k+1}(t_*) + \overline{\varphi}_{k-1}(t_*) - \varphi_{k-1}(t_*)\Big)
\end{aligned}
$$

which is greater than zero for N large enough, leading to a contradiction and hence concluding the proof. □

Proof of Proposition 3.1.1. Our goal is to estimate $|\psi^N(x_k,t) - \psi(x_k,t)|$. To do this, let us define the error function

$$
\boldsymbol{e}_k \;=\; \psi_k^N - \psi_k\,. \tag{3.8}
$$

Using a Taylor expansion, for $k \in \mathbb{T}_N$ there exist $c_k \in (x_k, x_{k+1})$ and $\widetilde{c}_k \in (x_{k-1}, x_k)$ such that

$$
\begin{aligned}
\psi_{k+1} \;&=\; \psi_k + \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi_k}{2!N^2} + \frac{\partial_x^3 \psi_k}{3!N^3} + \frac{\partial_x^4 \psi(c_k,t)}{4!N^4}\,, \\
\psi_{k-1} \;&=\; \psi_k - \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi_k}{2!N^2} - \frac{\partial_x^3 \psi_k}{3!N^3} + \frac{\partial_x^4 \psi(\widetilde{c}_k,t)}{4!N^4}\,.
\end{aligned}
$$

Adding the equations above we have that

$$
\psi_{k+1} + \psi_{k-1} \;=\; 2\psi_k + \frac{\partial_x^2 \psi_k}{N^2} + \frac{a_k}{N^4}\,, \tag{3.9}
$$

where $a_k = \frac{1}{4!}\big(\partial_x^4 \psi(c_k,t) + \partial_x^4 \psi(\widetilde{c}_k,t)\big)$. Since ψ is the solution of the PDE (2.5),

$$
\partial_x^2 \psi_k \;=\; \partial_t \psi_k + 2\partial_x\big(\psi_k \partial_x H_k\big) - e^{H_k} b(\psi_k) + e^{-H_k} d(\psi_k)\,,
$$

and replacing this into (3.9) gives us

$$\begin{aligned}
&N^2(\psi_{k+1}-2\psi_k+\psi_{k-1})-\frac{a_k}{N^2}\\
&=\ \partial_t\psi_k+2\partial_x\psi_k\partial_x H_k+2\psi_k\partial_x^2 H_k-e^{H_k}b(\psi_k)+e^{-H_k}d(\psi_k)\,. \qquad (3.10)
\end{aligned}$$

Observe that above we still have a first order derivative of ψ, which we want to write in terms of ψ_{k+1} and ψ_{k-1}. In order to do that, we apply again a Taylor expansion, telling us that, for $k\in\mathbb{T}_N$, there exist $d_k\in(x_k,x_{k+1})$ and $\widetilde{d}_k\in(x_{k-1},x_k)$ such that

$$\psi_{k+1}\ =\ \psi_k+\frac{\partial_x\psi_k}{N}+\frac{\partial_x^2\psi(d_k,t)}{2!N^2}\qquad\text{and}\qquad \psi_{k-1}\ =\ \psi_k-\frac{\partial_x\psi_k}{N}+\frac{\partial_x^2\psi(\widetilde{d}_k,t)}{2!N^2}\,.$$

Subtracting the equations above we have that

$$\psi_{k+1}-\psi_{k-1}\ =\ \frac{2}{N}\partial_x\psi_k+\frac{\overline{a}_k}{N^2},$$

where $\overline{a}_k=\frac{1}{2}(\partial_x^2\psi(d_k,t)-\partial_x^2\psi(\widetilde{d}_k,t))$. Substituting this into (3.10), we get

$$\begin{aligned}
\partial_t\psi_k\ &=N^2(\psi_{k+1}-2\psi_k+\psi_{k-1})-N(\psi_{k+1}-\psi_{k-1})\partial_x H_k-2\psi_k\partial_x^2 H_k\\
&\quad+e^{H_k}b(\psi_k)-e^{-H_k}d(\psi_k)-\frac{\overline{a}_k\partial_x H_k}{N}-\frac{a_k}{N^2}\,.
\end{aligned}$$

Recalling that ψ^N is a solution of (3.1) and the definition (3.8), we obtain that

$$\begin{aligned}
\partial_t\boldsymbol{e}_k\ &=N^2(\boldsymbol{e}_{k+1}-2\boldsymbol{e}_k+\boldsymbol{e}_{k-1})-N(\boldsymbol{e}_{k+1}-\boldsymbol{e}_{k-1})\partial_x H_k\\
&\quad-\Big[\frac{1}{2}\big(S_1^N+S_{-1}^N+2\big)\psi_k^N-2\psi_k\Big]\partial_x^2 H_k\\
&\quad+e^{H_k}\big(b(\psi_k^N)-b(\psi_k)\big)-e^{-H_k}\big(d(\psi_k^N)-d(\psi_k)\big)+\frac{\overline{a}_k\partial_x H_k}{N}+\frac{a_k}{N^2}\,.
\end{aligned}$$

Since

$$\Big|\frac{1}{2}\big(S_1^N+S_{-1}^N+2\big)\psi_k-2\psi_k\Big|\ \leq\ \frac{\|\partial_x\psi\|_\infty}{N}\,,$$

hence

$$\begin{aligned}
\partial_t\boldsymbol{e}_k\ &\leq N^2(\boldsymbol{e}_{k+1}-2\boldsymbol{e}_k+\boldsymbol{e}_{k-1})-N(\boldsymbol{e}_{k+1}-\boldsymbol{e}_{k-1})\partial_x H_k\\
&\quad-\Big[\frac{1}{2}\big(S_1^N+S_{-1}^N+2\big)\boldsymbol{e}_k\Big]\partial_x^2 H_k+e^{H_k}\big(b(\psi_k^N)-b(\psi_k)\big)-e^{-H_k}\big(d(\psi_k^N)-d(\psi_k)\big)\\
&\quad+\frac{\overline{a}_k\partial_x H_k}{N}+\frac{a_k}{N^2}+\frac{\|\partial_x\psi\|_\infty}{N}\partial_x^2 H_k\,.
\end{aligned}$$

Recalling (3.6), we get that

$$\begin{aligned}
\partial_t\boldsymbol{e}_k\ &\leq N^2(\boldsymbol{e}_{k+1}-2\boldsymbol{e}_k+\boldsymbol{e}_{k-1})-N(\boldsymbol{e}_{k+1}-\boldsymbol{e}_{k-1})\partial_x H_k\\
&\quad+C_*(|\boldsymbol{e}_{k+1}|+|\boldsymbol{e}_k|+|\boldsymbol{e}_{k-1}|+N^{-1})\,.
\end{aligned}$$

We have therefore proved that $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_N)$ is a subsolution for (3.3). Consider now $z_k(t) = \exp(\lambda C_* t)/N$, where $\lambda > 0$. Note that $z_k(t)$ does not depend on the space and a simple calculation permits to check that it will be a supersolution of (3.3) provided $\lambda > 3+\frac{1}{C_*}$. Fix henceforth some λ satisfying the condition above. By the Lemma 3.1.2 we have that

$$\boldsymbol{e}_k(t) \;\leq\; \frac{\exp(\lambda C_* t)}{N} \;\leq\; \frac{\exp(\lambda C_* T)}{N}\,.$$

Repeating the previous argument for $-\boldsymbol{e}_k$, we get that

$$|\boldsymbol{e}_k(t)| \;\leq\; \frac{\exp(\lambda C_* T)}{N}\,.$$

Therefore,

$$\sup_{t\in[0,T]}\max_{k\in\mathbb{T}_N} |\psi_k^N - \psi_k| \;\leq\; CN^{-1}\,,$$

finishing the proof. □

3.2 Dynkin Martingale

A natural way to obtain martingales from a particle system consists on the Dynkin's Formula which is quite suitable when we start with a function of the process: subtracting the initial value and the compensator gives us a martingale. In this section we will get the Dynkin martingale associated with the Markov process presented in Section 2.2.

From now on, we will use several times the notations

$$\Delta_N f(k) \;=\; N^2\Big[f\big(\tfrac{k+1}{N}\big)+f\big(\tfrac{k-1}{N}\big)-2f\big(\tfrac{k}{N}\big)\Big] \qquad \text{and} \tag{3.11}$$

$$\widetilde{\nabla}_N f(k) \;=\; \frac{N}{2}\Big[f\big(\tfrac{k+1}{N}\big)-f\big(\tfrac{k-1}{N}\big)\Big]\,. \tag{3.12}$$

Note that (3.11) is the discrete Laplacian while (3.12) is not the usual discrete derivative but it also approximates the continuous derivative in the case f is smooth. Equivalently, we may have defined the (perturbed) Markov process presented in Section 2.2 through its infinitesimal generator L_N which acts on functions $f:\Omega_N\to\mathbb{R}$ as

$$\begin{aligned}
\mathsf{L}_N f(\eta) \;=\; & \sum_{k\in\mathbb{T}_N} N^2\eta_k \exp\big\{H_{k+1}-H_k\big\}\Big[f(\eta^{k,k+1})-f(\eta)\Big]\\
&+\sum_{k\in\mathbb{T}_N} N^2\eta_k \exp\big\{H_{k-1}-H_k\big\}\Big[f(\eta^{k,k-1})-f(\eta)\Big]\\
&+\sum_{k\in\mathbb{T}_N} \ell b(\ell^{-1}\eta_k)\exp\big\{H_k\big\}\Big[f(\eta^{k,+})-f(\eta)\Big]\\
&+\sum_{k\in\mathbb{T}_N} \ell d(\ell^{-1}\eta_k)\exp\big\{-H_k\big\}\Big[f(\eta^{k,-})-f(\eta)\Big]\,,
\end{aligned}$$

where

$$\eta_j^{k,k\pm 1} \;=\; \begin{cases} \eta_j\,, & \text{if } j \neq k, k\pm 1 \\ \eta_k - 1\,, & \text{if } j = k \text{ and } \eta_k \geq 1 \\ \eta_{k\pm 1} + 1\,, & \text{if } j = k\pm 1 \text{ and } \eta_k \geq 1 \\ \eta_k\,, & \text{if } j = k \text{ and } \eta_k = 0 \\ \eta_{k\pm 1}\,, & \text{if } j = k\pm 1 \text{ and } \eta_k = 0 \end{cases}$$

and

$$\eta_j^{k,+} \;=\; \begin{cases} \eta_j\,, & \text{if } j \neq k \\ \eta_k + 1\,, & \text{if } j = k \end{cases}\,, \quad \eta_j^{k,-} \;=\; \begin{cases} \eta_j\,, & \text{if } j \neq k \\ \eta_k - 1\,, & \text{if } j = k \text{ and } \eta_k \geq 1 \\ \eta_k\,, & \text{if } j = k \text{ and } \eta_k = 0 \end{cases}\,.$$

It is a well known fact that the process $Z_f(t)$ defined by

$$M_f(t) \;=\; f(\eta(t)) - f(\eta(0)) - \int_0^t \mathsf{L}_N f(\eta(s))ds$$

is a martingale with respect to the natural filtration, the so-called *Dynkin martingale*, see [12, Appendix] for instance. Fix some $k \in \mathbb{T}_N$ and to simplify the notation we will omit the time variable in function H. Picking up the particular $f(\eta) = \eta_k$ tells us that

$$\begin{aligned} M_k(t) \;&= \eta_k(t) - \eta_k(0) - \int_0^t \Big[- N^2\eta_k(s)\Big[\exp\big\{H_{k+1} - H_k\big\} + \exp\big\{H_{k-1} - H_k\big\}\Big] \\ &\quad + N^2\eta_{k+1}(s)\exp\big\{H_k - H_{k+1}\big\} + N^2\eta_{k-1}(s)\exp\big\{H_k - H_{k-1}\big\} \\ &\quad + \ell b(\ell^{-1}\eta_k)\exp\big\{H_k\big\} - \ell d(\ell^{-1}\eta_k)\exp\big\{-H_k\big\}\Big]ds \end{aligned}$$

is a martingale. Since H has a finite Lipschitz constant, a Taylor expansion gives us that

$$\exp\big\{H_{k\pm 1} - H_k\big\} \;= 1 + H_{k\pm 1} - H_k + \frac{\big(H_{k\pm 1} - H_k\big)^2}{2!} + \mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k\pm 1}{N}, s\big)\,,$$

where the error term $\mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k\pm 1}{N}, s\big)$ is $O(N^{-3})$ uniform on $k \in \mathbb{T}_N$. This allows us to rewrite the above martingale as

$$\begin{aligned} M_k(t) \;&= \eta_k(t) - \eta_k(0) - \int_0^t \Big[N^2\big[\eta_{k+1}(s) + \eta_{k-1}(s) - 2\eta_k(s)\big] \\ &\quad - \eta_k(s)N^2\big[H_{k+1} + H_{k-1} - 2H_k\big] + \eta_{k+1}N^2\big(H_k - H_{k+1}\big) + \eta_{k-1}N^2\big(H_k - H_{k-1}\big) \\ &\quad + \ell b(\ell^{-1}\eta_k)\exp\big\{H_k\big\} - \ell d(\ell^{-1}\eta_k)\exp\big\{-H_k\big\} + \boldsymbol{A}_k(s)\Big]ds\,, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{A}_k(s) \;&= N^2\Big[\frac{1}{2}\big(H_{k+1} - H_k\big)^2\eta_{k+1}(s) + \frac{1}{2}\big(H_{k-1} - H_k\big)^2\eta_{k-1}(s) - \frac{1}{2}\big(H_{k+1} - H_k\big)^2\eta_k(s) \\ &\quad - \frac{1}{2}\big(H_{k-1} - H_k\big)^2\eta_k(s) + \mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k+1}{N}, s\big)\eta_{k+1}(s) + \mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k-1}{N}, s\big)\eta_{k-1}(s) \end{aligned}$$

$$- \mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k+1}{N}, s\big)\eta_k(s) - \mathbf{Err}\big(\tfrac{k}{N}, \tfrac{k-1}{N}, s\big)\eta_k(s)\Big] .$$

Using by Taylor that $H_{k\pm1} - H_k \; = \; \pm\frac{1}{N}\partial_x H_k + \frac{1}{2N^2}\partial_x^2 H_k + O(N^{-3})$, (3.11) and (3.12), we can rewrite the martingale simply as

$$\begin{aligned} M_k(t) \; = \; & \eta_k(t) - \eta_k(0) - \int_0^t \Big[\Delta_N \eta_k(s) - \eta_k(s)\Delta_N H_k - 2\widetilde{\nabla}_N \eta_k(s)\partial_x H_k \\ & - \frac{1}{2}\big(\eta_{k+1} + \eta_{k-1}\big)\partial_x^2 H_k + \ell b(\ell^{-1}\eta_k)\exp\big\{H_k\big\} - \ell d(\ell^{-1}\eta_k)\exp\big\{-H_k\big\} \\ & + \boldsymbol{A}_k + O(N^{-1})\eta_{k+1}(s) + O(N^{-1})\eta_{k-1}\Big] ds . \end{aligned}$$

Dividing the equation above by ℓ and using that the discrete Laplacian approximates the continuous Laplacian, it yields that

$$\begin{aligned} Z^N\big(t, \tfrac{k}{N}\big) \; = & \; X^N\big(t, \tfrac{k}{N}\big) - X^N\big(0, \tfrac{k}{N}\big) - \int_0^t \Big[\Delta_N X^N\big(s, \tfrac{k}{N}\big) - 2\widetilde{\nabla}_N X^N\big(s, \tfrac{k}{N}\big)\partial_x H_k \\ & - \frac{1}{2}\Big(X^N\big(s, \tfrac{k+1}{N}\big) + X^N\big(s, \tfrac{k-1}{N}\big) + 2X^N\big(s, \tfrac{k}{N}\big)\Big)\partial_x^2 H_k \\ & + b\big(X^N\big(s, \tfrac{k}{N}\big)\big)\exp\big\{H_k\big\} - d\big(X^N\big(s, \tfrac{k}{N}\big)\big)\exp\big\{-H_k\big\} + \boldsymbol{B}_k(s)\Big] ds , \end{aligned} \tag{3.13}$$

is a martingale for each $k \in \mathbb{T}_N$, now in a suitable form to our future purposes, where

$$\begin{aligned} \boldsymbol{B}_k(s) \; = & \; \frac{1}{2N^2}(\partial_x H_k)^2 \Delta_N X^N\big(s, \tfrac{k}{N}\big) \\ & + O(N^{-1})X^N\big(s, \tfrac{k+1}{N}\big) + O(N^{-1})X^N\big(s, \tfrac{k}{N}\big) + O(N^{-1})X^N\big(s, \tfrac{k-1}{N}\big) \end{aligned}$$

is the term which will not contribute in the limit as N goes to infinity, as we shall see later.

3.3 Duhamel's Principle

In this section we formulate a version of Duhamel's Principle for the martingales in (3.13), which will be necessary in the proof of Theorem 2.2.1. Let $T_N(t) = e^{\Delta_N t}$ the semigroup on $C(\mathbb{R}^{\mathbb{T}_N})$ generated by the discrete Laplacian Δ_N. To not overload notation, the spatial variable will be omitted in the sequel. Keeping this in mind, the martingales of (3.13) can be shortly written as

$$\begin{aligned} Z^N\big(t\big) \; = & \; X^N(t) - X^N(0) - \int_0^t \Big[\Delta_N X^N(s) - 2\widetilde{\nabla}_N X^N(s)\partial_x H(s) \\ & - \frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)X^N(s)\partial_x^2 H(s) \\ & + b\big(X^N(s)\big)\exp\big\{H(s)\big\} - d\big(X^N(s)\big)\exp\big\{-H(s)\big\} + \boldsymbol{B}(s)\Big] ds . \end{aligned} \tag{3.14}$$

Note that $Z^N(0) = 0$. Below, when we say that a stochastic process evolving on $\mathbb{R}^{\mathbb{T}_N}$ is a martingale, we mean that each one of its N coordinates are martingales.

Proposition 3.3.1 (Duhamel's Principle for $X^N(t)$). *Consider the martingale Z^N defined in* (3.14)*, which evolves on $\mathbb{R}^{\mathbb{T}_N}$. We have that*

$$\begin{aligned} X^N\big(t\big) \;&= T_N(t)X^N(0) + \int_0^t T_N(t-s)\Big[-2\widetilde{\nabla}_N X^N(s)\partial_x H(s) \\ &\quad -\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)X^N(s)\partial_x^2 H(s) + b\big(X^N(s)\big)\exp\big\{H(s)\big\} \\ &\quad - d\big(X^N(s)\big)\exp\big\{-H(s)\big\} + \boldsymbol{B}(s)\Big]ds + \int_0^t T_N(t-s)dZ^N(s)\,. \end{aligned} \tag{3.15}$$

Before proving the proposition above let us make a break to explain the meaning of the last integral on right hand-side of (3.15) and provide an integration by parts formula for it. Its definition is given by:

$$\int_0^t T_N(t-s)dZ^N(s) \;\stackrel{\text{def}}{=}\; \lim_{\Delta s_i\to 0}\sum_{i=1}^n T_N(t-s_i)\big[Z^N(s_i)-Z^N(s_{i-1})\big]\,, \tag{3.16}$$

where $\{0=s_0,\dots,s_n=t\}$ is a partition of the interval $[0,t]$ and Δs_i is its size. Expanding the summation on the rigth side above, we get

$$\begin{aligned} \sum_{i=1}^n T_N(t-s_i)\big[Z^N(s_i)-Z^N(s_{i-1})\big] \;&=\; \sum_{i=1}^n T_N(t-s_i)Z^N(s_i) - \sum_{i=0}^{n-1} T_N(t-s_{i+1})Z^N(s_i) \\ &=\; \sum_{i=1}^{n-1}\big[T_N(t-s_i)-T_N(t-s_{i+1})\big]Z^N(s_i) + Z^N(t)\,. \end{aligned}$$

Now dividing and multiplying the sum on the right hand-side of the equation above by $(s_{i+1}-s_i)$ and then taking the limit as $\Delta s_i \to 0$, one can deduce that

$$\int_0^t T_N(t-s)dZ^N(s) \;=\; \int_0^t \partial_t T_N(t-s)Z^N(s)\,ds + Z^N(t) - Z^N(0)\,.$$

Due to $T_N(t)=e^{\Delta_N t}$, we obtain that

$$\int_0^t T_N(t-s)dZ^N(s) \;=\; \int_0^t \Delta_N T_N(t-s)Z^N(s)\,ds + Z^N(t) - Z^N(0)\,, \tag{3.17}$$

which is the desired integration-by-parts formula.

Proof of the Proposition 3.3.1. We show that (3.15) and (3.14) are equivalent. Below we will calculate the integral of the Laplacian of X^N to ease some later calculations. Recall (3.17). Then,

$$\begin{aligned} \int_0^t \Delta_N X^N(s)ds &= \int_0^t \Delta_N T_N(s)X^N(0)\,ds \\ &\quad + \int_0^t\int_0^s \Delta_N T_N(s-v)\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v) - \frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)X^N(v)\partial_x^2 H(v) \\ &\quad + b\big(X^N(v)\big)\exp\big\{H(v)\big\} - d\big(X^N(v)\big)\exp\big\{-H(v)\big\} + \boldsymbol{B}(v)\Big]dv\,ds \end{aligned}$$

$$+\int_0^t\int_0^s \Delta_N^2 T_N(s-v)Z^N(v)\,dv\,ds+\int_0^t \Delta_N Z^N(s)\,ds\,.$$

Again by $T_N(t)=e^{\Delta_N t}$ and by Fubini's Theorem, we obtain

$$\begin{aligned}\int_0^t \Delta_N X^N(s)\,ds &= T_N(t)X^N(0)-X^N(0)\\ &+\int_0^t T_N(t-v)\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v)-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(v)\partial_x^2 H(v)\\ &+b\big(X^N(v)\big)\exp\big\{H(v)\big\}-d\big(X^N(v)\big)\exp\big\{-H(v)\big\}+\boldsymbol{B}(v)\Big]dv\\ &-\int_0^t\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v)-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(v)\partial_x^2 H(v)\\ &+b\big(X^N(v)\big)\exp\big\{H(v)\big\}-d\big(X^N(v)\big)\exp\big\{-H(v)\big\}+\boldsymbol{B}(v)\Big]dv\\ &+\int_0^t \Delta_N T_N(t-v)Z^N(v)\,dv\,.\end{aligned}$$

Using the above result we have that

$$\begin{aligned}X^N(t) &= X^N(0)+T_N(t)X^N(0)-X^N(0)+\int_0^t T_N(t-v)\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v)\\ &-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(v)\partial_x^2 H(v)+b\big(X^N(v)\big)\exp\big\{H(v)\big\}\\ &-d\big(X^N(v)\big)\exp\big\{-H(v)\big\}+\boldsymbol{B}(v)\Big]dv-\int_0^t\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v)\\ &-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(v)\partial_x^2 H(v)+b\big(X^N(v)\big)\exp\big\{H(v)\big\}\\ &-d\big(X^N(v)\big)\exp\big\{-H(v)\big\}+\boldsymbol{B}(v)\Big]dv+\int_0^t \Delta_N T_N(t-v)Z^N(v)dv\\ &+\int_0^t\Big[-2\widetilde{\nabla}_N X^N(s)\partial_x H(s)-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(s)\partial_x^2 H(s)\\ &+b\big(X^N(s)\big)\exp\big\{H(s)\big\}-d\big(X^N(s)\big)\exp\big\{-H(s)\big\}+\boldsymbol{B}(s)\Big]ds+Z^N(t).\end{aligned}$$

Therefore

$$\begin{aligned}X^N(t)=T_N(t)X^N(0)&+\int_0^t T_N(t-v)\Big[-2\widetilde{\nabla}_N X^N(v)\partial_x H(v)\\ &-\frac{1}{2}\Big(S_1^N+S_{-1}^N+2\Big)X^N(v)\partial_x^2 H(v)+b\big(X^N(v)\big)\exp\big\{H(v)\big\}\\ &-d\big(X^N(v)\big)\exp\big\{-H(v)\big\}+\boldsymbol{B}(v)\Big]dv+\int_0^t T_N(t-v)dZ^N(v)\,.\end{aligned}$$

□

The next proposition provides an expression for the solution of the ODE system (3.1).

Proposition 3.3.2 (Duhamel's Principle for $\psi^N(t)$)**.** *The solution* $\psi^N(t)$ *of* (3.1) *solves*

$$\begin{aligned}\psi_k^N(t) &= T_N(t)\psi_k^N(0) + \int_0^t T_N(t-s)\Big[-\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\psi_k^N(s)\partial_x^2 H_k(s) \\ &\quad - 2\widetilde{\nabla}_N \psi_k^N(s)\partial_x H_k(s) + b(\psi_k^N(s))\exp\big\{H_k(s)\big\} - d(\psi_k^N(s))\exp\big\{-H_k(s)\big\}\Big]ds\end{aligned} \tag{3.18}$$

with $k = 1, \dots, N$.

Proof. We will differentiate the right hand side of (3.18) in order to verify that it is, in fact, a solution for the ODE system (3.1). It gives us

$$\begin{aligned}&\frac{\partial}{\partial t}T_N(t)\psi_k^N(0) + \int_0^t \frac{\partial}{\partial t}T_N(t-s)\Big[-\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\psi_k^N(s)\partial_x^2 H_k(s) \\ &- 2\widetilde{\nabla}_N \psi_k^N(s)\partial_x H_k(s) + b(\psi_k^N(s))\exp\big\{H_k(s)\big\} - d(\psi_k^N(s))\exp\big\{-H_k(s)\big\}\Big]ds \\ &+ T_N(0)\Big[-\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\psi_k^N(t)\partial_x^2 H_k(t) - 2\widetilde{\nabla}_N \psi_k^N(t)\partial_x H_k(t) \\ &+ b(\psi_k^N(t))\exp\big\{H_k(t)\big\} - d(\psi_k^N(t))\exp\big\{-H_k(t)\big\}\Big].\end{aligned}$$

By the definition of the semigroup T_N and the fact that the Laplacian is linear, the expression above is equal to

$$\begin{aligned}&\Delta_N\Bigg[T_N(t)\psi_k^N(0) + \int_0^t T_N(t-s)\Big[-\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\psi_k^N(s)\partial_x^2 H_k(s) \\ &- 2\widetilde{\nabla}_N \psi_k^N(s)\partial_x H_k(s) + b(\psi_k^N(s))\exp\big\{H_k(s)\big\} - d(\psi_k^N(s))\exp\big\{-H_k(s)\big\}\Big]ds\Bigg] \\ &- \frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\psi_k^N(t)\partial_x^2 H_k(t) - 2\widetilde{\nabla}_N \psi_k^N(t)\partial_x H_k(t) \\ &+ b(\psi_k^N(t))\exp\big\{H_k(t)\big\} - d(\psi_k^N(t))\exp\big\{-H_k(t)\big\},\end{aligned}$$

so the equation (3.18) holds true. □

3.4 High density limit

In this section we prove Theorem 2.2.1. Before going through details, let us explain the involved ideas. Noting the resemblance of (3.15) and (3.18), we would like to have that

$$\sup_{t\in[0,T]} \|Y^N(t)\|_\infty \to 0 \tag{3.19}$$

in some sense, where

$$Y^N(t) = \int_0^t T_N(t-s)dZ^N(s)$$

is the only (random) term which differs (3.15) from (3.18). Since the solution $\psi^N(t)$ of the semi-discrete scheme converges to the solution of the concerning PDE (see Section 3.1), Gronwall inequality would finish the job, assuring that $X^N(t)$ converges to the solution of the PDE (2.5). However, (3.19) is not true, or at least, not clear how to be argued. The reason of this is the following: the essential ingredient to prove that a process as Y^N goes to zero as $N \to \infty$ requires that the corresponding martingale $Z^N(t)$ is bounded, which is not true.

Next, we mixture ideas from the original strategy of [4] with the approach of [8]. Instead of working with $X^N(t)$, we will deal with a stopped process $\overline{X}^N(t)$ close to $X^N(t)$. Fix $\varepsilon_0 > 0$, let

$$\tau \;=\; \inf\big\{t : \|X^N(t) - \psi^N(t)\|_\infty > \varepsilon_0\big\}$$

and define

$$\overline{X}^N(t) \;=\; \begin{cases} X^N(t), & \text{if } t \leq \tau \\ W^N(t), & \text{if if } t > \tau \end{cases}$$

where $W^N(t) = \big(W_1^N(t), \dots, W_N^N(t)\big)$ is defined for each $k \in \mathbb{T}_N$ and $t > \tau$ through

$$\begin{cases} \partial_t W_k^N = N^2\big(W_{k+1}^N - 2W_k^N + W_{k-1}^N\big) - N\big(W_{k+1}^N - W_{k-1}^N\big)\partial_x H_k \\ \qquad\qquad -\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)W_k^N \partial_x^2 H_k + e^{H_k} b\big(W_k^N\big) - e^{-H_k} d\big(W_k^N\big)\,, \\ W_k^N(\tau) = X_k^N(\tau)\,, \quad k \in \mathbb{T}_N\,. \end{cases}$$

Note that W^N is defined as in 3.1 but considering the density X^N as an initial condition to ensure that $\overline{X}^N$ does not have jumps.

The reason we may work with $\overline{X}^N(t)$ instead of $X^N(t)$ is that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|\overline{X}^N(t) - \psi^N(t)\|_\infty \;=\; 0 \;\text{ a.s.} \tag{3.20}$$

implies

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|X^N(t) - \psi^N(t)\|_\infty \;=\; 0 \;\text{ a.s.,}$$

since if $t \leq \tau$, $\overline{X}^N(t) = X^N(t)$ and if $t > \tau$, $W_k^N(t) = \psi_k^N(t)$ for each $k \in \mathbb{T}_N$.

Thus, our goal now is to prove (3.20). Of course, there are many different choices for the stopped process. Consider $\overline{X}_k^N(\cdot) = \overline{X}^N(\cdot, k/N)$. The main features of $\overline{X}^N(t)$ are the following. First, its version of Duhamel's Principle is given by

$$\begin{aligned} \overline{X}_k^N(t) \;=\; & T_N(t)\overline{X}_k^N(0) + \overline{Y}_k^N(t) + \int_0^t T_N(t-s)\Big[-\frac{1}{2}\Big(S_1^N + S_{-1}^N + 2\Big)\overline{X}_k^N(s)\partial_x^2 H_k(s) \\ & - 2\widetilde{\nabla}_N \overline{X}_k^N(s)\partial_x H_k(s) + b\Big(\overline{X}_k^N(s)\Big)\exp\big\{H_k(s)\big\} - d\Big(\overline{X}_k^N(s)\Big)\exp\big\{-H_k(s)\big\}\Big]ds \end{aligned} \tag{3.21}$$

where

$$\overline{Y}^N(t) \;=\; \int_0^t T_N(t-s)d\overline{Z}^N(s\wedge\tau)\,,$$

with $\overline{Z}^N$ being the same martingale given in (3.14) changing X^N by $\overline{X}^N$. The proof of (3.21) above is similar to the proof of Proposition 3.3.1 and will be omitted. Second, but not less important, is the fact that there exists some $\boldsymbol{C}>0$ such that

$$\sup_{t\in[0,T]} \|\overline{X}^N(t)\|_\infty \;\leq\; \boldsymbol{C} \tag{3.22}$$

for all large enough $N\in\mathbb{N}$. The inequality above can be argued as follows. Since the solution ψ of the PDE (2.5) is smooth and defined on a compact domain, it is bounded. Proposition 3.1.1 tells us that ψ^N converges uniformly to ψ, hence ψ^N is bounded as well by some constant $c_1>0$. By the definition of the stopping time τ, the process $\overline{X}^N(t)$ is bounded by $c_1+\varepsilon_0$ for any time $t<\tau$. After time τ, the process runs deterministically under the same dynamics of ψ^N, but with the random initial condition given by $\overline{X}^N(\tau)$ at time τ. Since $\|\overline{X}^N(\tau)\|_\infty \leq c_1+\varepsilon_0+\frac{1}{\ell}$, an argument on super-solutions (similar to that one presented in Section 3.1) gives that $\overline{X}^N(t)$ is also bounded for some constant for all time $t>\tau$.

To furnish the necessary martingales for the proof of Theorem 2.2.1 and in many others cases, we provide a general statement in the next proposition. This is a well-known result. However we could not find any reference in the literature in a suitable form. For this reason, we include it here for sake of completeness.

Proposition 3.4.1. *Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain taking values on the countable set Ω. Denote by $\lambda:\Omega\times\Omega\to\mathbb{R}_+$ the jump rate, assume that $\lambda(x,x)=0$ for all $x\in\Omega$ and*

$$\sup_{x\in\Omega}\Big\{\sum_{y\in\Omega}\lambda(x,y)\Big\} \;<\; \infty\,.$$

This continuous time Markov chain can described as follows. When at the state $x\in\Omega$, the next state is chosen according to the minimum of a family of independent exponentials of parameter $\lambda(x,z)$, where $z\in\Omega$, $z\neq x$. If the minimum of such exponentials is attained at the exponential of parameter $\lambda(x,y)$, the process remains at x during a period of time equal to the value of this exponential and then jumps to y. Denote by $N_t(x,y)$ the number of times the process has made the transition from x to y in the time interval $[0,t]$. Then

$$\mathcal{M}_t \;=\; N_t(x,y)-\lambda(x,y)\int_0^t \mathbb{1}_{[X_s=x]}\,ds$$

is a martingale with respect to the natural filtration.

Proof. Denote by μ the initial distribution and by $\mathcal{F}_t$ the natural filtration, i.e. the σ-algebra generated by the process until time $t\geq 0$. Let $0\leq u\leq t$.

$$\begin{aligned}\mathbb{E}_\mu\Big[N_t(x,y)-\lambda(x,y)\int_0^t \mathbb{1}_{[X_s=x]}ds\Big|\mathcal{F}_u\Big] &= N_u(x,y)-\lambda(x,y)\int_0^u \mathbb{1}_{[X_s=x]}ds\\ &\quad+\mathbb{E}_\mu\Big[N_t(x,y)-N_u(x,y)-\lambda(x,y)\int_u^t \mathbb{1}_{[X_s=x]}ds\Big|\mathcal{F}_u\Big].\end{aligned}$$

By the Markov Property, in order to show is null the second parcel in the r.h.s. of the equation above, it is sufficient to proof that

$$\mathbb{E}_z\Big[N_t(x,y)-\lambda(x,y)\int_0^t \mathbb{1}_{[X_s=x]}ds\Big] \;=\; 0 \tag{3.23}$$

for any $z\in\Omega$ and any $t\geq 0$. Let $0=t_0<t_1<\cdots<t_n=t$ be a partition of the interval $[0,t]$. Expression (3.23) can be rewritten as

$$\sum_{i=0}^{n-1}\mathbb{E}_z\Big[N_{t_{i+1}}(x,y)-N_{t_i}(x,y)+\lambda(x,y)\int_{t_i}^{t_{i+1}}\mathbb{1}_{[X_s=x]}ds\Big]\,.$$

Since the probability of two or more jumps in a interval of length h is $O(h^2)$, it is enough to show that

$$\mathbb{E}_z\Big|N_{t_{i+1}}(x,y)-N_{t_i}(x,y)-\lambda(x,y)\int_{t_i}^{t_{i+1}}\mathbb{1}_{[X_{t_i}=x]}ds\Big| \;=\; O\big((t_{i+1}-t_i)^2\big)\,.$$

By the Markov Property, it is enough to assure that $\mathbb{E}_x|N_h(x,y)-\lambda(x,y)h|$ is $O(h^2)$. On his hand, this is a consequence of the definition of $N_h(x,y)$. □

Denote $\delta f(t)=f(t)-f(t^-)$. Applying the above proposition considering our model, we have:

Lemma 3.4.2. *The following processes are martingales with respect to the natural filtration, for all* $k=0,\ 1,\cdots,\ N-1$*,*

$$\begin{aligned}\mathcal{M}_t^{N,1} &= \ell\big[\overline{X}_k^N(t)-\overline{X}_k^N(0)\big]-\int_0^t \ell N^2\Big[\overline{X}_{k-1}^N(s)e^{H_k-H_{k-1}}-2\overline{X}_k^N(s)e^{H_{k+1}-H_k}\\ &\quad+\overline{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}}\Big]ds-\int_0^t \ell\Big[b(\overline{X}_k^N(s))e^{H_k}-d(\overline{X}_k^N(s))e^{-H_k}\Big]ds\,, \qquad (3.24)\\ \mathcal{M}_t^{N,2} &= \ell^2\sum_{s\leq t}\big(\delta\overline{X}_k^N(s)\big)^2-\int_0^t \ell N^2\Big[\overline{X}_{k-1}^N(s)e^{H_k-H_{k-1}}+2\overline{X}_k^N(s)e^{H_{k+1}-H_k}\\ &\quad+\overline{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}}\Big]ds-\int_0^t \ell\Big[b(\overline{X}_k^N(s))e^{H_k}+d(\overline{X}_k^N(s))e^{-H_k}\Big]ds\,, \qquad (3.25)\\ \mathcal{M}_t^{N,3} &= -\ell^2\sum_{s\leq t}\delta\overline{X}_k^N(s)\,\delta\overline{X}_{k+1}^N(s)-\int_0^t \ell N^2\Big[\overline{X}_k^N(s)e^{H_{k+1}-H_k}+\overline{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}}\Big]ds\,. \qquad (3.26)\end{aligned}$$

Proof. As we shall see below, each of the expressions (3.24), (3.25), and (3.26) are equal to the number of times some kind of transitions has been made minus the integral in time of the corresponding rates.

In (3.24), the parcel

$$\ell\big[\overline{X}_k^N(t)-\overline{X}_k^N(0)\big]$$

of that expression counts how many times in $[0,t]$ the Markov process $(\eta_t)_{t\geq 0}$ has made a transition $\eta_k=j$ to $\eta_k=j+1$ for some $j\in\mathbb{N}$, minus how many times the process has made a transition $\eta_k=j+1$ to $\eta_k=j$, normalized by the parameter ℓ.

In (3.25), the parcel

$$\ell^2 \sum_{s\leq t} (\delta \overline{X}_k^N(s))^2$$

of that expression counts how many times in $[0, t]$ the process has made a transition $\eta_k = j$ to $\eta_k = j \pm 1$ for some $j \in \mathbb{N}$.

In (3.26), the parcel

$$-\ell^2 \sum_{s\leq t} \delta \overline{X}_k^N(s)\, \delta \overline{X}_{k+1}^N(s)$$

of that expression counts how many times in $[0, t]$ particles have jumped between the sites k and $k+1$. Since the integral parts in (3.24), (3.25) and (3.26) are the integrals in time of the respective rates, one application of Proposition 3.4.1 finishes the proof. □

As we shall see, (3.21) and (3.22) are two necessary ingredients in the proof of (3.20), as well as the next lemma.

Lemma 3.4.3. *Recall the constant $\boldsymbol{C} > 0$ as in* (3.22)*. Then, there exists some $a = a(\boldsymbol{C}, T) > 0$ such that, for any $\varepsilon > 0$,*

$$\mathbb{P}\left[e^{-4T} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\right] \;\leq\; 4N^3 \exp(-a\varepsilon^2 \ell)\,. \tag{3.27}$$

The demonstration of Lemma 3.4.3 follows in a similar way to the demonstration of Lemma 4.10 in [4] before proving it, we need the following Lemma 3.4.4 and recall two results of [4], which are the main ingredients of its proof. Denote

$$\nabla_N^+ f(k) \;=\; N\Big[f\big(\tfrac{k+1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big] \quad \text{and} \quad \nabla_N^- f(k) \;=\; N\Big[f\big(\tfrac{k-1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big]\,.$$

Lemma 3.4.4. *The process*

$$\sum_{s\leq t} \big(\delta\langle \overline{Z}^N(t), \varphi\rangle\big)^2 - (N\ell)^{-1} \int_0^t \Big\langle \overline{X}^N(s) e^{\nabla_N^+ H/N}, (\nabla_N^+\varphi)^2 + (\nabla_N^-\varphi)^2 \Big\rangle\, ds$$
$$- (N\ell)^{-1} \int_0^t \Big\langle b(\overline{X}^N(s)) e^{H} + d(\overline{X}^N(s)) e^{-H}, \varphi^2 \Big\rangle\, ds,$$

where $\langle \cdot, \cdot\rangle$ is the inner product in $L^2(\mathbb{T})$ with respect to the Lebesgue measure, is a mean zero martingale with respect to the natural filtration.

Proof. First, note that the process $\overline{X}^N$ and $\overline{Z}^N$ have the same jumps of discontinuity. Then given $\varphi \in S^N$ we have that

$$\begin{aligned}\sum_{s\leq t} \big(\delta\langle \overline{Z}^N(t), \varphi\rangle\big)^2 \;&=\; \sum_{s\leq t} \frac{1}{N^2} \Big(\sum_{k=0}^{N-1} \varphi_k \delta \overline{X}_k^N(s)\Big)^2 \\ &=\; \sum_{s\leq t} \frac{1}{N^2} \sum_{k=0}^{N-1} \varphi_k^2 \big(\delta \overline{X}_k^N(s)\big)^2 + \sum_{s\leq t} \frac{2}{N^2} \sum_{k=0}^{N-1} \varphi_k \varphi_{k+1} \delta \overline{X}_k^N(s) \delta \overline{X}_{k+1}^N(s),\end{aligned}$$

so, by (3.25) and (3.26), the process below is a martingale:

$$
\begin{aligned}
&\sum_{s\le t}\big(\delta\langle \overline{Z}^N(t),\varphi\rangle\big)^2 \\
&\quad-\sum_{k=0}^{N-1}\int_0^t \frac{\varphi_k^2}{\ell}\Big(\overline{X}^N_{k-1}(s)e^{H_k-H_{k-1}}+2\overline{X}^N_k(s)e^{H_{k+1}-H_k}+\overline{X}^N_{k+1}(s)e^{H_{k+2}-H_{k+1}}\Big)\\
&\qquad\qquad+\frac{\varphi_k^2}{N^2\ell}\Big(b(\overline{X}^N_k(s))e^{H_k}+d(\overline{X}^N_k(s))e^{-H_k}\Big)\,ds\\
&\quad+\sum_{k=0}^{N-1}\int_0^t \frac{2\varphi_k\varphi_{k+1}}{\ell}\Big(\overline{X}^N_k(s)e^{H_{k+1}-H_k}+\overline{X}^N_{k+1}(s)e^{H_{k+2}-H_{k+1}}\Big)\,ds\,. \qquad (3.28)
\end{aligned}
$$

Note now that

$$
\sum_{k=0}^{N-1}\frac{\varphi_k^2}{N}\Big(b(\overline{X}^N_k(s))e^{H_k}+d(\overline{X}^N_k(s))e^{-H_k}\Big) \;=\; \Big\langle b(\overline{X}^N(s))e^{H}+d(\overline{X}^N(s))e^{-H},\varphi^2\Big\rangle, \qquad (3.29)
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\Big[\varphi_k^2\big(\overline{X}^N_{k-1}(s)e^{H_k-H_{k-1}}+2\overline{X}^N_k(s)e^{H_{k+1}-H_k}+\overline{X}^N_{k+1}(s)e^{H_{k+2}-H_{k+1}}\big)\\
&\qquad-2\varphi_k\varphi_{k+1}\big(\overline{X}^N_k(s)e^{H_{k+1}-H_k}+\overline{X}^N_{k+1}(s)e^{H_{k+2}-H_{k+1}}\big)\Big]\\
&\quad=\sum_{k=0}^{N-1}\overline{X}^N_k(s)e^{H_{k+1}-H_k}\big((\varphi_{k+1}-\varphi_k)^2+(\varphi_{k-1}-\varphi_k)^2\big)\\
&\quad=\sum_{k=0}^{N-1}\frac{\overline{X}^N_k(s)e^{H_{k+1}-H_k}}{N^2}\big((\nabla_N^+\varphi_k)^2+(\nabla_N^-\varphi_k)^2\big)\\
&\quad=N^{-1}\Big\langle \overline{X}^N(s)e^{\nabla_N^+H/N},(\nabla_N^+\varphi)^2+(\nabla_N^-\varphi)^2\Big\rangle\,. \qquad (3.30)
\end{aligned}
$$

Substituting (3.29) and (3.30) in (3.28) we conclude that

$$
\begin{aligned}
\sum_{s\le t}\big(\delta\langle \overline{Z}^N(t),\varphi\rangle\big)^2-(N\ell)^{-1}\int_0^t\Big\langle \overline{X}^N(s)e^{\nabla_N^+H/N},(\nabla_N^+\varphi)^2+(\nabla_N^-\varphi)^2\Big\rangle\,ds\\
-(N\ell)^{-1}\int_0^t\Big\langle b(\overline{X}^N(s))e^{H}+d(\overline{X}^N(s))e^{-H},\varphi^2\Big\rangle\,ds
\end{aligned}
$$

is a mean zero martingale. □

Lemma 3.4.5 (Lemma 4.3 in [4]). *Let $f=N\mathbb{1}_{[k/N,(k+1)/N)}$. Then exist function h_N such that*

$$
\big\langle\big(\nabla_N^+T_N(t)f\big)^2+\big(\nabla_N^-T_N(t)f\big)^2+\big(T_N(t)f\big)^2,1\big\rangle\le h_N(t) \qquad (3.31)
$$

where $\int_0^t h_N(s)ds\le CN+t$.

Lemma 3.4.6 (Lemma 4.4 in [4]). *Let $m(t)$ be a bounded martingale of finite variation defined on $[t_0, t_1]$ with $m(t_0) = 0$ and satisfying:*

i) m *is right-continuous with left limits.*

ii) $|\delta m(t)| \leq 1$ *for* $t_0 \leq t \leq t_1$.

iii) $\sum_{t_0 \leq s \leq t} (\delta m(s))^2 - \int_{t_0}^t g(s)ds$ *is a mean* 0 *martingale with* $0 \leq g(s) \leq h(s)$*, where* $h(s)$ *is a bounded deterministic function and* $g(s)$ *is adapted to the natural filtration.*

Then

$$\mathbb{E} \exp\big(m(t_1)\big) \;\leq\; \exp\left(\frac{3}{2}\int_{t_0}^{t_1} h(s)ds\right).$$

Proof of Lemma 3.4.3. Fix $\bar{t} \in (0, T]$, $k \in \mathbb{T}_N$ and consider $f = N\mathbb{1}_{[k/N,(k+1)/N)}$. Define

$$m(t) \;=\; \Big\langle \int_0^t T_N(\bar{t} - s)d\overline{Z}^N(s), f \Big\rangle, \quad \text{for all } 0 \leq t \leq \bar{t}\,.$$

which satisfies $m(\bar{t}) = \overline{Y}^N(\bar{t}, k/N)$. Since $\overline{Z}^N$ is a (vector) martingale, then the integral $\int_0^t T_N(\bar{t} - s)d\overline{Z}^N(s)$ is a zero mean (vector) martingale, hence $m(t)$ is a zero mean martingale on $0 \leq t \leq \bar{t}$ as well. By the integration by parts formula (3.17), the discontinuity jumps of the process $m(t)$ are the same discontinuity jumps of $\langle \overline{Z}^N(t), T_N(\bar{t} - t)f \rangle$. Therefore, by Lemma 3.4.4,

$$\begin{aligned}\sum_{s \leq t} \big(\delta m(s)\big)^2 - (N\ell)^{-1} \int_0^t \Big\langle \overline{X}^N(s) e^{\nabla_N^+ H/N}, \big(\nabla_N^+ T_N(\bar{t} - s)f\big)^2 + \big(\nabla_N^- T_N(\bar{t} - s)f\big)^2 \Big\rangle ds \\ - (N\ell)^{-1} \int_0^t \Big\langle b(\overline{X}^N(s))e^H + d(\overline{X}^N(s))e^{-H}, \big(T_N(\bar{t} - s)f\big)^2 \Big\rangle ds \qquad (3.32)\end{aligned}$$

is a mean 0 martingale. For $\theta \in [0, 1]$, consider $\theta\ell m(t)$ instead of $m(t)$. Recall the constant $\boldsymbol{C} > 0$ as in (3.22), and rewrite the martingale above as

$$(\theta\ell)^2 \sum_{s \leq t} (\delta m(s))^2 - (\theta\ell)^2 \int_0^t g(s)ds\,.$$

Since $\overline{X}^N(s)e^{\nabla_N^+ H/N}$ and $b(\overline{X}^N(s))e^H + d(\overline{X}^N(s))e^{-H}$ are bounded in modulus by a constant $\overline{a}(\mathbf{C})$ and recalling Lemma 3.4.5, we have that

$$(\theta\ell)^2 g(s) \;\leq\; \overline{a}(\boldsymbol{C})\theta^2 \ell N^{-1} h_N(t)\,.$$

So, by Lemma 3.4.6,

$$\mathbb{E}\big[\exp(\theta\ell m(t))\big] \;\leq\; \exp\left(\frac{3}{2}\,\overline{a}(\boldsymbol{C})\theta^2 \ell N^{-1} \int_0^t h_N(s)ds\right) \;\leq\; \exp\big(\overline{a}(\boldsymbol{C})\theta^2\ell(1 + tN^{-1})\big)\,. \qquad (3.33)$$

Fix $\varepsilon > 0$. Using Chebychev inequality we obtain that

$$\mathbb{P}\big[\overline{Y}^N(\bar{t}, k/N) > \varepsilon\big] \leq \mathbb{E}\big[\exp(\theta\ell\overline{Y}^N(\bar{t}, k/N))\big]\exp(-\theta\ell\varepsilon) = \mathbb{E}\big[\exp(\theta\ell m(\bar{t}))\big]\exp(-\theta\ell\varepsilon)\,.$$

Since $\bar{t} \leq T$, we may assume that $\bar{t}/N \leq 1$, then by (3.33)

$$\mathbb{P}\big[\overline{Y}^N(\bar{t}, k/N) > \varepsilon\big] \;\leq\; \exp\big(\theta\ell(\bar{a}(\boldsymbol{C})\theta - \varepsilon)\big) \;=\; \exp(-\ell\varepsilon^2 a(\boldsymbol{C}))\,,$$

with $a(\boldsymbol{C})$ depending on $\bar{a}(\boldsymbol{C})$, ε and θ. Arguing analogously for $\mathbb{P}\big[\overline{Y}^N(\bar{t}, k/N) < -\varepsilon\big]$, we conclude that, for $0 < \bar{t} < T$ and $k \in \mathbb{T}_N$,

$$\mathbb{P}\Big[\,\big|\overline{Y}^N(\bar{t}, k/N)\big| > \varepsilon\Big] \;\leq\; 2\exp(-\ell\varepsilon^2 a(\boldsymbol{C}))\,,$$

and taking the supremum in k, we then get

$$\mathbb{P}\Big[\,\big\|\overline{Y}^N(\bar{t}, \cdot)\big\|_\infty > \varepsilon\Big] \;\leq\; 2N\exp(-\ell\varepsilon^2 a(\boldsymbol{C}))\,. \tag{3.34}$$

Integrating the laplacian of $\overline{Y}^N$ in (3.17) and using Fubini's Theorem, we deduce that

$$\int_0^t \Delta_N \overline{Y}^N(s)ds \;=\; \overline{Y}^N(t) - \overline{Z}^N(t).$$

Then, for $nTN^{-2} \leq t \leq (n+1)TN^{-2}$ with $n = 0, \ldots, N^2 - 1$,

$$\int_{nTN^{-2}}^t \Delta_N \overline{Y}^N(s)ds \;=\; \overline{Y}^N(t) - \overline{Y}^N(nTN^{-2}) - \overline{Z}^N(t) + \overline{Z}^N(nTN^{-2}).$$

So, taking the supremum norm and recalling the definition of the discrete Laplacian,

$$\|\overline{Y}^N(t)\|_\infty \;\leq\; \|\overline{Y}^N(nTN^{-2})\|_\infty + 4N^2\int_{nTN^{-2}}^t \|\overline{Y}^N(s)\|_\infty ds + \|\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\|_\infty.$$

Using Gronwall's inequality and taking the supremum in the time we get that

$$\begin{aligned}\sup_{[nTN^{-2},(n+1)TN^{-2}]} &\|\overline{Y}^N(t)\|_\infty \\ &\leq \Big(\|\overline{Y}^N(nTN^{-2})\|_\infty + \sup_{[nTN^{-2},(n+1)TN^{-2}]}\|\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\|_\infty\Big)e^{4T}.\end{aligned} \tag{3.35}$$

Observe that $\delta\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big) = \delta\overline{Z}^N(t) = \delta\overline{X}^N(t)$. Then, by Lemma 3.4.2, for k fixed and $\theta \in [0,1]$,

$$\begin{aligned}(\theta\ell)^2 \sum_{nTN^{-2}\leq s\leq t} \big(\delta\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big)\big)^2 - \theta^2\ell\int_{nTN^{-2}}^t N^2\Big[\overline{X}_{k-1}^N(s)e^{H_k - H_{k-1}} \\ + 2\overline{X}_k^N(s)e^{H_{k+1}-H_k} + \overline{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}}\Big] + \Big[b(\overline{X}_k^N(s))e^{H_k} + d(\overline{X}_k^N(s))e^{-H_k}\Big]ds\,,\end{aligned}$$

is a mean zero martingale for $nTN^{-2} \leq t \leq (n+1)TN^{-2}$. Again recalling the constant $\boldsymbol{C}$ as in (3.22), we rewrite the martingale above as

$$(\theta\ell)^2 \sum_{nTN^{-2}\leq s\leq t} \Big(\delta\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big)\Big)^2 - \theta^2\ell \int_{nTN^{-2}}^{t} N^2\overline{g}(s)ds\,,$$

and by Lemma 3.4.6, we have that

$$\mathbb{E}\bigg[\exp\Big(\theta\ell\big(\overline{Z}^N((n+1)TN^{-2}) - \overline{Z}^N(nTN^{-2})\big)\Big)\bigg] \;\leq\; \exp\big(\overline{a}(\boldsymbol{C})\theta^2\ell T\big)\,.$$

Fix $\varepsilon > 0$. Using the Doob's inequality we obtain that

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{[nTN^{-2},(n+1)TN^{-2}]} \big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big) > \varepsilon\Big]\\
&\quad= \mathbb{P}\Big[\sup_{[nTN^{-2},(n+1)TN^{-2}]} \exp\Big(\theta\ell\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big)\Big) > \exp(\theta\ell\varepsilon)\Big]\\
&\quad\leq \mathbb{E}\bigg[\exp\Big(\theta\ell\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big)\Big)\bigg]\exp(-\theta\ell\varepsilon)\\
&\quad\leq \exp\big(\overline{a}(\boldsymbol{C})\theta^2\ell T - \theta\ell\varepsilon\big) = \exp\big(-a(\boldsymbol{C},T)\ell\varepsilon^2\big)\,.
\end{aligned}$$

Repeating analogously to $\mathbb{P}\Big[\sup_{[nTN^{-2},(n+1)TN^{-2}]}\big(\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big) < -\varepsilon\Big]$ and taking the supremum norm we have that

$$\mathbb{P}\Big[\sup_{[nTN^{-2},(n+1)TN^{-2}]} \big\|\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\big\|_\infty > \varepsilon\Big] \;=\; 2N\exp\big(-a(\boldsymbol{C},T)\ell\varepsilon^2\big)\,. \tag{3.36}$$

Thus, by (3.35)

$$\begin{aligned}
&\mathbb{P}\bigg[e^{-4T}\sup_{[nTN^{-2},(n+1)TN^{-2}]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\bigg]\\
&\quad\leq \mathbb{P}\big[\|\overline{Y}^N(nTN^{-2})\|_\infty > \varepsilon\big] + \mathbb{P}\bigg[\sup_{[nTN^{-2},(n+1)TN^{-2}]} \|\overline{Z}^N(t) - \overline{Z}^N(nTN^{-2})\|_\infty > \varepsilon\bigg],
\end{aligned}$$

and by (3.34) and (3.36)

$$\mathbb{P}\bigg[e^{-4T}\sup_{[nTN^{-2},(n+1)TN^{-2}]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\bigg] \;\leq\; 4N\exp\big(-a(\boldsymbol{C},T)\ell\varepsilon^2\big).$$

Note that

$$\mathbb{P}\bigg[e^{-4T}\sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\bigg] \;\leq\; \sum_{n=0}^{N^2-1}\mathbb{P}\bigg[e^{-4T}\sup_{[nTN^{-2},(n+1)TN^{-2}]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\bigg],$$

therefore

$$\mathbb{P}\bigg[e^{-4T}\sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\bigg] \;\leq\; 4N^3\exp\big(-a(\boldsymbol{C},T)\ell\varepsilon^2\big),$$

concluding the proof. □

Corollary 3.4.7. *Consider* $\overline{Y}^N(t) = \int_0^t T_N(t)(t-s)d\overline{Z}^N(s)$. *Then*

$$N^{4\|\partial_x H\|_\infty/\pi} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty \to 0 \ \ a.s.$$

if $\dfrac{N^{4\|\partial_x H\|^2_\infty/\pi^2}\log N}{\ell} \to 0$ *when* $N \to \infty$.

Proof. By Lemma 3.4.3,

$$\mathbb{P}\left[e^{-4T} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\right] \ \le \ 4N^3 \exp(-a\varepsilon^2\ell)\,,$$

therefore

$$\mathbb{P}\left[e^{-4T} N^{4\|\partial_x H\|_\infty/\pi} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\right] \ \le \ 4N^3 \exp\left(\frac{-a\varepsilon^2\ell}{N^{4\|\partial_x H\|^2_\infty/\pi^2}}\right).$$

By hypothesis $c\log(N)N^{4\|\partial_x H\|^2_\infty/\pi^2} \ < \ \ell$, for any c constant and N large enough. Then

$$\sum_{N=1}^{\infty} 4N^3 \exp\left(\frac{-a\varepsilon^2\ell}{N^{4\|\partial_x H\|^2_\infty/\pi^2}}\right) \ < \ \sum_{N=1}^{\infty} \frac{1}{N^{1+\delta}} \ < \ \infty\,.$$

So we have that

$$\sum_{N=1}^{\infty} \mathbb{P}\left[e^{-4T} N^{4\|\partial_x H\|_\infty/\pi} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty > \varepsilon\right] < \infty\,.$$

And by Borel-Cantelli's Lemma,

$$N^{4\|\partial_x H\|_\infty/\pi} \sup_{[0,T]} \|\overline{Y}^N(t)\|_\infty \to 0 \ \text{ almost surely},$$

and we conclude the proof. □

Consider S^N the real-valued step function on $[0,1]$ that is constant on the intervals $[kN^{-1},(k+1)N^{-1})$ with $0 \le k \le N-1$. Before proving Theorem 2.2.1 we need to define an orthonormal basis to space $\big(S^N, \langle\cdot,\cdot\rangle\big)$. If N is odd integer define $\varphi_{0,N} \equiv 1$ and for m even, with $0 \le m \le N-1$, $\varphi_{m,N}(r) = \sqrt{2}\cos(\pi mkN^{-1})$ and $\phi_{m,N}(r) = \sqrt{2}\sin(\pi mkN^{-1})$ for $r \in [kN^{-1},(k+1)N^{-1})$. So $\{\varphi_{m,N}, \phi_{m,N}\}$ are eigenvectors of Δ_N, with eigenvalues defined by $-\beta_{m,N} = -2N^2(1-\cos(\pi mN^{-1}))$, and is the basis sought. If N is even, we added the eigenvector $\varphi_{N,N}(r) = \cos(\pi k)$.

We define the base as in [4], where this fact has not been proven, we include here the proof for completeness.

Lemma 3.4.8. *The terms of the sequence* $\{\varphi_{m,N}, \phi_{m,N}\}_m$ *are eigenvectors of* Δ_N, *with eigenvalues* $-\beta_{m,N}$ *and form an orthonormal basis for* $\big(S^N, \langle\cdot,\cdot\rangle\big)$.

Proof. To show that $\varphi_{m,N}$ are eigenvectors note that, for m fixed and $r \in [kN^{-1}, (k+1)N^{-1}$ we have that

$$\Delta_N \varphi_{m,N}(r) \;=\; 2\sqrt{2}N^2 \cos(\pi m k N^{-1})\big(\cos(\pi m N^{-1}) - 1\big) \;=\; -\beta_{m,N}\varphi_{m,N}(r).$$

For $\phi_{m,N}$ we have that

$$\Delta_N \phi_{m,N}(r) \;=\; 2\sqrt{2}N^2\big(\sin(\pi m k N^{-1})\cos(\pi m N^{-1}) - \sin(\pi m k N^{-1})\big) \;=\; -\beta_{m,N}\varphi_{m,N}(r).$$

If N is odd we have N eigenvectors and therefore $\{\varphi_{m,N}, \phi_{m,N}\}_m$ forms a basis. In the case of N even we have $N-1$ eigenvectors and for this reason we add $\varphi_{N,N}(r) = \cos(\pi k)$ to form the basis. To show that it is orthonormal note that

$$\langle \varphi_{m,N}, \phi_{m,N}\rangle \;=\; \sum_{k=0}^{N-1} \frac{2}{N}\cos(\pi m k N^{-1})\sin(\pi m k N^{-1}) \;=\; \sum_{k=0}^{N-1}\frac{1}{N}\sin(2\pi m k N^{-1}) \;=\; 0\,,$$

since the terms in the last sum cancel each other out. Thus we conclude the proof. □

Lemma 3.4.9. *Given $g \in S^N$, for $r \in [0,1]$,*

$$T_N(t)g(r) \;=\; \sum_{\substack{m\in[0,N-1];\\ m \text{ even}}} e^{-\beta_{m,N}t}\big(\langle g, \varphi_{m,N}\rangle\varphi_{m,N}(r) + \langle g, \phi_{m,N}\rangle\phi_{m,N}(r)\big). \tag{3.37}$$

Proof. Note that, since $\{\varphi_{m,N}, \phi_{m,N}\}_m$ is a basis for S^N, g can be written as

$$g(r) = \sum_{\substack{m\in[0,N-1];\\ m \text{ even}}} \langle g, \varphi_{m,N}\rangle\varphi_{m,N}(r) + \langle g, \phi_{m,N}\rangle\phi_{m,N}(r).$$

Then,

$$\begin{aligned} T_N(t)g(r) \;&=\; \sum_{j=0}^{\infty}\frac{\Delta_N^j t^j}{j!}\Big(\sum_m \langle g, \varphi_{m,N}\rangle\varphi_{m,N}(r) + \langle g, \phi_{m,N}\rangle\phi_{m,N}(r)\Big)\\ &=\; \sum_{j=0}^{\infty}\sum_m \frac{-\beta_{m,N}^j\varphi_{m,N}(r)}{j!}t^j\langle g, \varphi_{m,N}\rangle - \frac{\beta_{m,N}^j\phi_{m,N}(r)}{j!}t^j\langle g, \phi_{m,N}\rangle\\ &=\; \exp\big(-\beta_{m,N}t\big)\sum_m \langle g, \varphi_{m,N}\rangle\varphi_{m,N} + \langle g, \phi_{m,N}\rangle\phi_{m,N}. \end{aligned}$$

And we concluded the proof. □

Now we can prove the high density limit for the perturbed process.

Proof of Theorem 2.2.1. We want to evaluate $\sup_{[0,T]}\|\overline{X}^N(t) - \psi(t)\|_\infty$. To do so, we first consider Proposition 3.1.1 and then we can calculate $\sup_{[0,T]}\|\overline{X}^N(t) - \psi^N(t)\|_\infty$. Denote $\boldsymbol{e}^N(t) := \overline{X}^N(t) - \psi^N(t)$. Using the expressions (3.15) for X^N and (3.18) for ψ^N, we have that

$$
\begin{aligned}
\|\boldsymbol{e}^N(t)\|_\infty \leq \big\|T_N(t)\boldsymbol{e}^N(0)\big]\big\|_\infty + \bigg\| \int_0^t T_N(t-s) d\overline{Z}^N(s) \bigg\|_\infty \\
+ \bigg\| \int_0^t T_N(t-s)\Big[-2\widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) - \frac{1}{2}\big(S_1^N + S_{-1}^N + 2\big)\boldsymbol{e}^N(s)\partial_x^2 H(s) \\
+ e^{H(s)}\Big(b(\overline{X}^N(s) - b(\psi_k^N(s))\Big) - e^{-H(s)}\Big(d(\overline{X}^N(s)) - d(\psi_k^N(s))\Big) + \boldsymbol{B}(s)\Big] ds \bigg\|_\infty .
\end{aligned}
$$

Note that $\frac{1}{2}\|(S_1^N + S_{-1}^N + 2)\boldsymbol{e}^N\|_\infty \leq 2\|\boldsymbol{e}^N\|_\infty$ and, as T_N is a contraction, we also have to $\|T_N(t)\boldsymbol{e}^N(0)\|_\infty \leq \|\boldsymbol{e}^N(0)\|_\infty$. Moreover, consider $\overline{C}$ constant such that

$$
\overline{C} \overset{\text{def}}{=} \max\Big\{\|e^H\|_\infty \cdot \|b\|_L, \|e^{-H}\|_\infty \cdot \|d\|_L\Big\}, \tag{3.38}
$$

where $\|b\|_L$ and $\|d\|_L$ are the Lispchitz constants of the functions b and d, respectively. Then

$$
\begin{aligned}
\|\boldsymbol{e}^N(t)\|_\infty \leq \|\boldsymbol{e}^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \bigg\| \int_0^t 2T_N(t-s)\widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) ds \bigg\|_\infty \\
+ \int_0^t 2\big\|\boldsymbol{e}^N(s)\big\|_\infty \big\|\partial_x^2 H(s)\big\|_\infty ds + \int_0^t 2\overline{C}\big\|\boldsymbol{e}^N(s)\big\|_\infty ds + \int_0^t \big\|\boldsymbol{B}(s)\big\|_\infty ds .
\end{aligned} \tag{3.39}
$$

Observe the third term of the above inequality separately. We will use that

$$
\widetilde{\nabla}_N\big[\boldsymbol{e}^N(s)\partial_x H(s)\big] = \widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) + \boldsymbol{e}^N(s)\widetilde{\nabla}_N \partial_x H(s) .
$$

Therefore,

$$
\begin{aligned}
\bigg\|2\int_0^t T_N(t-s)\widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) ds\bigg\|_\infty \leq \bigg\|2\int_0^t T_N(t-s)\widetilde{\nabla}_N\big[\boldsymbol{e}^N(s)\partial_x H(s)\big] ds\bigg\|_\infty \\
+ \bigg\|2\int_0^t T_N(t-s)\boldsymbol{e}^N(s)\widetilde{\nabla}_N \partial_x H(s) ds\bigg\|_\infty .
\end{aligned}
$$

Since $T_N(t)\nabla_N = \nabla_N T_N(t)$ and T_N contraction, it yields that

$$
\begin{aligned}
&\bigg\|2\int_0^t T_N(t-s)\widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) ds\bigg\|_\infty \\
&\leq 2\int_0^t \big\|\widetilde{\nabla}_N T_N(t-s)\big[\boldsymbol{e}^N(s)\partial_x H(s)\big]\big\|_\infty ds + \int_0^t \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds .
\end{aligned} \tag{3.40}
$$

Using (3.37) we then have that

$$
\begin{aligned}
&\widetilde{\nabla}_N T_N(t-s)\big[\boldsymbol{e}^N(s)\partial_x H(s)\big] \\
&= \widetilde{\nabla}_N \sum_m e^{-\beta_{m,N}(t-s)}\big(\langle \boldsymbol{e}^N(s)\partial_x H(s), \varphi_{m,N}\rangle \varphi_{m,N} + \langle \boldsymbol{e}^N(s)\partial_x H(s), \phi_{m,N}\rangle \phi_{m,N}\big)
\end{aligned}
$$

$$= \sum_m e^{-\beta_{m,N}(t-s)}\big(\langle \boldsymbol{e}^N(s)\partial_x H(s), \varphi_{m,N}\rangle \widetilde{\nabla}_N \varphi_{m,N} + \langle \boldsymbol{e}^N(s)\partial_x H(s), \phi_{m,N}\rangle \widetilde{\nabla}_N \phi_{m,N}\big)\,.$$

By the definition of $\varphi_{m,N}$ e $\phi_{m,N}$ there exist a constant c such that

$$\big|\widetilde{\nabla}_N \varphi_{m,N} - (-\pi m \phi_{m,N})\big| \leq \frac{c}{N} \quad \text{and} \quad \big|\widetilde{\nabla}_N \phi_{m,N} - \pi m \varphi_{m,N}\big| \leq \frac{c}{N}\,.$$

Thus,

$$\begin{aligned}
&2\int_0^t \big\|\widetilde{\nabla}_N T_N(t-s)\big[\boldsymbol{e}^N(s)\partial_x H(s)\big]\big\|_\infty ds \leq 2\int_0^t \sum_m e^{-\beta_{m,N}(t-s)} \\
&\left\|\langle \boldsymbol{e}^N(s)\partial_x H(s), \varphi_{m,N}\rangle\left(\frac{c}{N} - \pi m \phi_{m,N}\right) + \langle \boldsymbol{e}^N(s)\partial_x H(s), \phi_{m,N}\rangle\left(\frac{c}{N} + \pi m \varphi_{m,N}\right)\right\|_\infty ds \\
&\leq 2\int_0^t \sum_m e^{-\beta_{m,N}(t-s)}\Big(\|\langle \boldsymbol{e}^N(s)\partial_x H(s), \varphi_{m,N}\rangle\|_\infty + \|\langle \boldsymbol{e}^N(s)\partial_x H(s), \phi_{m,N}\rangle\|_\infty\Big)\frac{c}{N} ds \\
&\quad + 2\int_0^t \sum_m e^{-\beta_{m,N}(t-s)} \pi m\big(\|\langle \boldsymbol{e}^N(s)\partial_x H(s), \varphi_{m,N}\rangle \phi_{m,N}\|_\infty \\
&\qquad\qquad + \|\langle \boldsymbol{e}^N(s)\partial_x H(s), \phi_{m,N}\rangle \varphi_{m,N}\|_\infty\big) ds\,.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality and writing the definition of $\beta_{m,N}$ we have that

$$\begin{aligned}
&2\int_0^t \big\|\widetilde{\nabla}_N T_N(t-s)\big[\boldsymbol{e}^N(s)\partial_x H(s)\big]\big\|_\infty ds \\
&\quad \leq \frac{4c}{N}\int_0^t \sum_m \exp\big[-2N^2(1-\cos(\pi m N^{-1}))(t-s)\big]\|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds \\
&\qquad + 4\int_0^t \sum_m \exp\big[-2N^2(1-\cos(\pi m N^{-1}))(t-s)\big]\pi m \|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds\,.
\end{aligned}$$

Note that

$$\sum_m \exp\big[-2N^2(1-\cos(\pi m N^{-1}))(t-s)\big] \leq N.$$

By Taylor's expansion $1-\cos(\pi m N^{-1}) \geq \dfrac{\pi^2 m^2}{2N^2} + O(N^{-3})$, we get that

$$\begin{aligned}
&2\int_0^t \big\|\widetilde{\nabla}_N T_N(t-s)\big[\boldsymbol{e}^N(s)\partial_x H(s)\big]\big\|_\infty ds \leq 4c\int_0^t \|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds \\
&+ 4\pi\int_0^t \sum_m \exp\left[-2N^2\left(\frac{\pi^2 m^2}{2N^2} + O(N^{-3})\right)(t-s)\right] m \|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds\,.
\end{aligned}$$

Coming back to (3.40) we have that

$$\left\|2\int_0^t T_N(t-s)\widetilde{\nabla}_N \boldsymbol{e}^N(s)\partial_x H(s) ds\right\|_\infty$$

$$\leq \int_0^t \big(4c\|\partial_x H(s)\|_\infty + \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty\big)\|\boldsymbol{e}^N(s)\|_\infty ds$$
$$+ 4\pi \int_0^t \sum_m \exp\big[-\big(\pi^2 m^2 + O(N^{-1})\big)(t-s)\big] m \|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds\,.$$

To conclude the proof of the theorem, let us return to the inequality (3.39).

$$\|\boldsymbol{e}^N(t)\|_\infty \;\leq\; \|\boldsymbol{e}^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \int_0^t \Big(2\|\partial_x^2 H(s)\|_\infty + 2\overline{C}\Big)\|\boldsymbol{e}^N(s)\|_\infty ds$$
$$+ \int_0^t \big\|\boldsymbol{B}(s)\big\|_\infty + \int_0^t \big(4c\|\partial_x H(s)\|_\infty + \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty\big)\|\boldsymbol{e}^N(s)\|_\infty ds$$
$$+ 4\pi \int_0^t \sum_m \exp\big[-\big(\pi^2 m^2 + O(N^{-1})\big)(t-s)\big] m \|\partial_x H(s)\|_\infty \|\boldsymbol{e}^N(s)\|_\infty ds$$

and applying Gronwall's inequality, we get that

$$\|\boldsymbol{e}^N(t)\|_\infty \;\leq \Bigg(\|\boldsymbol{e}^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \int_0^t \big\|\boldsymbol{B}(s)\big\|_\infty ds\Bigg)$$
$$\times \exp\Bigg\{\int_0^t 2\|\partial_x^2 H(s)\|_\infty + 2\overline{C} + 4c\|\partial_x H(s)\|_\infty + \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty$$
$$+ 4\pi \sum_m \exp\big[-\big(\pi^2 m^2 + O(N^{-1})\big)(t-s)\big] m \|\partial_x H(s)\|_\infty \, ds\Bigg\}\,.$$

Note that

$$\int_0^t 4\pi \sum_m \exp\big[-\big(\pi^2 m^2 + O(N^{-1})\big)(t-s)\big] m \|\partial_x H(s)\|_\infty \, ds$$
$$\leq \; 4\|\partial_x H\|_\infty \sum_m \frac{1 - \exp\big[-\big(\pi^2 m^2 + O(N^{-1})\big)t\big]}{\pi m}$$
$$\leq \; \frac{4\|\partial_x H\|_\infty}{\pi} \sum_m \frac{1}{m} \;\leq\; \frac{4\|\partial_x H\|_\infty}{\pi} \log N\,.$$

Then,

$$\|\boldsymbol{e}^N(t)\|_\infty \;\leq\; \Bigg(\|\boldsymbol{e}^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \int_0^t \big\|\boldsymbol{B}(s)\big\|_\infty ds\Bigg)$$
$$\times \exp\Bigg\{\int_0^t 2\|\partial_x^2 H(s)\|_\infty + 2\overline{C} + 4c\|\partial_x H(s)\|_\infty + \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty \, ds\Bigg\} N^{4\|\partial_x H\|_\infty/\pi}\,.$$

Taking

$$\mathcal{C} \stackrel{\text{def}}{=} \exp\Bigg\{\int_0^t 2\|\partial_x^2 H(s)\|_\infty + 2\overline{C} + 4c\|\partial_x H(s)\|_\infty + \|\widetilde{\nabla}_N \partial_x H(s)\|_\infty \, ds\Bigg\},$$

we have that

$$\|\boldsymbol{e}^N(t)\|_\infty \;\leq\; \Bigg(\|\boldsymbol{e}^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \int_0^t \big\|\boldsymbol{B}(s)\big\|_\infty ds\Bigg)\mathcal{C} N^{4\|\partial_x H\|_\infty/\pi}\,.$$

Moreover,

$$
\begin{aligned}
\|\overline{X}^N(0) - \psi(0)\|_\infty &\leq \left|\frac{\eta_x(0)}{\ell} - \psi(0,x)\right| = \left|\frac{\lfloor \ell\psi(0,x)\rfloor}{\ell} - \psi(0,x)\right| \\
&= \frac{1}{\ell}\Big|\lfloor \ell\psi(0,x)\rfloor - \ell\psi(0,x)\Big| \leq \frac{1}{\ell},
\end{aligned}
$$

thus $\|\boldsymbol{e}^N(0)\|_\infty \mathcal{C} N^{4\|\partial_x H\|_\infty/\pi} \to 0$ almost surely as $N \to \infty$, and by Lemma 3.4.7 we conclude the proof. □

Chapter 4

Large Deviations

We begin this chapter by finding an expression for the Radon-Nikodym derivative which will be used directly to obtain the rate function of the large deviations. In the following sections we obtain the large deviations, starting with the upper bound, then the lower bound considering smooth profiles and then the lower bound for more general profiles.

4.1 Radon-Nikodym derivative

An important ingredient in the proof of large deviations consists in obtaining a law of large numbers for a class of perturbed processes. To find the rate function we need to calculate the Radon-Nikodym derivative $\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}$ where $\mathbb{P}_N$ and $\mathbb{P}_N^H$ are measures induced by processes considering $H \equiv 0$ and a general $H \in C^{1,2}$, respectively. This is the content of the next proposition.

Proposition 4.1.1 (An expression for the Radon-Nikodym derivative)**.** *Considering the model described above, the Radon-Nikodym derivative restricted to $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$ is given by*

$$\begin{aligned} \frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg|_{\mathcal{F}_t} = \exp\Bigg\{ - \ell N \Bigg[\int_0^t \frac{1}{N} \sum_{k=0}^{N-1} \Big[b\big(X_k^N(s)\big)\big(1 - e^{H_k}\big) + d\big(X_k^N(s)\big)\big(1 - e^{-H_k}\big) \\ - X_k^N(s)\Big(\Delta_N H_k + \frac{1}{2}\Big(\big(\nabla_N^+ H_k\big)^2 + \big(\nabla_N^- H_k\big)^2\Big) + O(1/N)\Big)\Big] ds \\ + \frac{1}{N}\sum_{k=0}^{N-1}\Big(H_k(t)X_k^N(t) - H_k(0)X_k^N(0) - \int_0^t X_k^N(s)\partial_s H_k ds\Big)\Bigg]\Bigg\}. \end{aligned} \tag{4.1}$$

In particular, we can write

$$\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg|_{\mathcal{F}_t} = \exp\Big\{ - \ell N \Big[J_H(X^N) + O(1/N)\Big]\Big\}, \tag{4.2}$$

where

$$
\begin{aligned}
J_H(u) = \int_0^t \int_{\mathbb{T}} \Big[& b\big(u(s,y)\big)\big(1-e^{H(s,y)}\big) + d\big(u(s,y)\big)\big(1-e^{-H(s,y)}\big) \\
& - u(s,y)\Big(\Delta H(s,y) + \big(\nabla H(s,y)\big)^2\Big)\Big]\, dy\, ds \\
+ \int_{\mathbb{T}} \Big[& H(t,y)u(t,y) - H(0,y)u(0,y) - \int_0^t u(s,y)\partial_s H(s,y)\, ds\Big] dy\,.
\end{aligned}
$$

Now we are in position to prove the Proposition 4.1.1 which is the basis for deriving the rate function of large deviations. The large deviations provide the speed of convergence of the process, which will be exponential, and this rate function will provide the exponent of this exponential function. To do so, we need the following general result which can be found in [12, Appendix 1, page 320].

Proposition 4.1.2. *Let P and $\overline{P}$ be the probability measures corresponding to two continuous time Markov chains on some countable space E, with bounded waiting times λ and $\overline{\lambda}$, respectively, and with transition probabilities p and $\overline{p}$, respectively. Assume that p and $\overline{p}$ vanish at the diagonal, that is, $p(x,x) = \overline{p}(x,x) = 0$ for all $x \in E$. Assume that P is absolutely continuous with respect to $\overline{P}$. Then, the Radon-Nikodym derivative of P with respect to $\overline{P}$ restricted to $\mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t)$ is given by*

$$
\frac{dP}{d\overline{P}}\bigg|_{\mathcal{F}_t}(X) \;=\; \exp\Bigg\{ - \int_0^t \lambda(X(s)) - \overline{\lambda}(X(s))ds + \sum_{s\leq t} \log\Bigg(\frac{\lambda\big(X(s)\big)p\big(X(s_-),X(s)\big)}{\overline{\lambda}\big(X(s)\big)\overline{p}\big(X(s_-),X(s)\big)}\Bigg)\Bigg\}, \tag{4.3}
$$

where X denotes a pure jump càdlàg time trajectory on E.

In the case of our work, $P = \mathbb{P}_N$ and $\overline{P} = \mathbb{P}_N^H$. The probabilities $\mathbb{P}_N$ and $\mathbb{P}_N^H$ are associated to trajectories $\eta(t)$ of course. However, recalling the definition (2.1), we will often write $X^N(\frac{k}{N},t)$ instead of $\ell^{-1}\eta_k(t)$, which makes notation shorter and enlightens ideas. Furthermore, recall the notation $H_k = H(\frac{k}{N},t) = H(\frac{k}{N},t_-)$, where this last equality holds since H is assumed to be smooth, in space and time, and write for simplicity $X^N(t) = X^N(\cdot,t)$.

For fixed N, we have that

$$
\begin{aligned}
\lambda(X^N(t)) \;&=\; \sum_{k=0}^{N-1} \ell\Big[b\big(X_k^N(t)\big) + d\big(X_k^N(t)\big) + 2N^2X_k^N(t)\Big]\,, \\
\overline{\lambda}(X^N(t)) \;&=\; \sum_{k=0}^{N-1} \ell\Big[b\big(X_k^N(t)\big)e^{H_k} + d\big(X_k^N(t)\big)e^{-H_k} + N^2X_k^N(t)e^{-H_k}\Big(e^{H_{k+1}} + e^{H_{k-1}}\Big)\Big]\,,
\end{aligned} \tag{4.4}
$$

$$p\big(X^N(s_-),X^N(s)\big) \;=\; \begin{cases} \ell b\big(X_k^N(s_-)\big)/\lambda(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)+1; \\ \ell d\big(X_k^N(s_-)\big)/\lambda(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1; \\ N^2\ell X_k^N(s_-)/\lambda(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1 \\ \qquad\qquad \text{and } \eta_{k+1}(s)=\eta_{k+1}(s_-)+1; \\ N^2\ell X_k^N(s_-)/\lambda(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1 \\ \qquad\qquad \text{and } \eta_{k-1}(s)=\eta_{k-1}(s_-)+1; \end{cases} \tag{4.5}$$

and

$$\overline{p}\big(X^N(s_-),X^N(s)\big) \;=\; \begin{cases} \ell b\big(X_k^N(s_-)\big)e^{H_k}/\overline{\lambda}(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)+1; \\ \ell d\big(X_k^N(s_-)\big)e^{-H_k}/\overline{\lambda}(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1; \\ N^2\ell X_k^N(s_-)e^{H_{k+1}-H_k}/\overline{\lambda}(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1 \\ \qquad\qquad \text{and } \eta_{k+1}(s)=\eta_{k+1}(s_-)+1; \\ N^2\ell X_k^N(s_-)e^{H_{k-1}-H_k}/\overline{\lambda}(X^N(s_-)), \text{ if } \eta_k(s)=\eta_k(s_-)-1 \\ \qquad\qquad \text{and } \eta_{k-1}(s)=\eta_{k-1}(s_-)+1. \end{cases} \tag{4.6}$$

Proof of Proposition 4.1.1. Given a path $\eta(t)$, define the sets of times

$$\begin{aligned} B_t^k &= \big\{s\le t : \eta_k(s)=\eta_k(s_-)+1\big\}\,, \\ D_t^k &= \big\{s\le t : \eta_k(s)=\eta_k(s_-)-1\big\}\,, \\ J_t^{k,k+1} &= \big\{s\le t : \eta_k(s)=\eta_k(s_-)-1 \text{ and } \eta_{k+1}(s)=\eta_{k+1}(s_-)+1\big\}\,, \\ J_t^{k,k-1} &= \big\{s\le t : \eta_k(s)=\eta_k(s_-)-1 \text{ and } \eta_{k-1}(s)=\eta_{k-1}(s_-)+1\big\}\,. \end{aligned}$$

Note that B_t^k and D_t^k represents the set of times at which some particle is created and destroyed at the site k, respectively. The set $J_t^{k,k+1}$ represents the times at which some particle jump the site k for site $k+1$ and $J_t^{k,k-1}$ the times at which some particle jump the site k for site $k-1$. Invoking Proposition 4.1.2, the expressions (4.4), (4.5), (4.6) and the sets defined above, we deduce that

$$\begin{aligned} \frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg|_{\mathcal{F}_t} = \exp\Bigg\{ -\int_0^t \sum_{k=0}^{N-1} \ell\Big[& b\big(X_k^N(s)\big)\big(1-e^{H_k}\big) + d\big(X_k^N(s)\big)\big(1-e^{-H_k}\big) \\ & + N^2 X_k^N(s)\Big(2-e^{H_{k+1}-H_k}-e^{H_{k-1}-H_k}\Big)\Big]\,ds \\ + \sum_{k=0}^{N-1}\Bigg(\sum_{s\in B_t^k}(-H_k) + \sum_{s\in D_t^k} H_k &+ \sum_{s\in J_t^{k,k+1}}(H_k-H_{k+1}) + \sum_{s\in J_t^{k,k-1}}(H_k-H_{k-1})\Bigg)\Bigg\}\,. \end{aligned}$$

Since H is smooth, by a Taylor expansion on the exponential function,

$$\begin{aligned} & 2-e^{H_{k+1}-H_k}-e^{H_{k-1}-H_k} \\ & \quad = -H_{k+1}+H_k-\frac{1}{2!}\big(H_{k+1}-H_k\big)^2 - H_{k-1}+H_k-\frac{1}{2!}\big(H_{k-1}-H_k\big)^2 + O(1/N^3)\,, \end{aligned}$$

hence

$$\begin{aligned} & N^2 X_k^N(s)\Big(2 - e^{H_{k+1}-H_k} - e^{H_{k-1}-H_k}\Big)\Big) \\ &= -X_k^N(s)\bigg(\Delta_N H_k + \frac{1}{2}\Big(\big(\nabla_N^+ H_k\big)^2 + \big(\nabla_N^- H_k\big)^2\Big) + O(1/N)\bigg) . \end{aligned}$$

Moreover,

$$\begin{aligned} & \sum_{s\in B_t^k}(-H_k) + \sum_{s\in D_t^k} H_k + \sum_{s\in Jt^{k,k+1}} (H_k - H_{k+1}) + \sum_{s\in J_t^{k,k-1}} (H_k - H_{k-1}) \\ &= \int_0^t (-H_k)\, dB_t^k + \int_0^t H_k\, dD_t^k + \int_0^t (H_k - H_{k+1})\, dJ_t^{k,k+1} + \int_0^t (H_k - H_{k-1})\, dJ_t^{k,k-1} \\ &= -\int_0^t H_k\, \big(dB_t^k - dD_t^k - dJ_t^{k,k+1} + dJ_t^{k-1,k} - dJ_t^{k,k-1} + dJ_t^{k+1,k}\big) \;=\; -\int_0^t H_k\, d\eta_k(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg|_{\mathcal{F}_t} &= \exp\bigg\{ -\ell N\bigg[\int_0^t \frac{1}{N}\sum_{k=0}^{N-1}\Big[b\big(X_k^N(s)\big)\big(1-e^{H_k}\big) + d\big(X_k^N(s)\big)\big(1-e^{-H_k}\big) \\ & - X_k^N(s)\bigg(\Delta_N H_k + \frac{1}{2}\Big(\big(\nabla_N^+ H_k\big)^2 + \big(\nabla_N^- H_k\big)^2\Big) + O(1/N)\bigg)\bigg] ds + \frac{1}{\ell N}\sum_{k=0}^{N-1}\int_0^t H_k\, d\eta_k(t)\bigg]\bigg\}. \end{aligned}$$

Applying integration by parts for Stieltjes measures (see for instance [6, Exercise 6.4, page 470]) and relation (2.1), we are lead to

$$\begin{aligned} \frac{1}{\ell N}\int_0^t H_k\, d\eta_k(t) &= \frac{1}{\ell N}\bigg[H_k(t)\eta_k(t) - H_k(0)\eta_k(0) - \int_0^t \eta_k(s)\partial_s H_k ds\bigg] \\ &= \frac{1}{N}\bigg[H_k(t)X_k^N(t) - H_k(0)X_k^N(0) - \int_0^t X_k^N(s)\partial_s H_k ds\bigg] . \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg|_{\mathcal{F}_t} &= \exp\bigg\{ -\ell N\bigg[\int_0^t \frac{1}{N}\sum_{k=0}^{N-1}\Big[b\big(X_k^N(s)\big)\big(1-e^{H_k}\big) + d\big(X_k^N(s)\big)\big(1-e^{-H_k}\big) \\ & \quad - X_k^N(s)\bigg(\Delta_N H_k + \frac{1}{2}\Big(\big(\nabla_N^+ H_k\big)^2 + \big(\nabla_N^- H_k\big)^2\Big) + O(1/N)\bigg)\bigg] ds \\ & \quad + \frac{1}{N}\sum_{k=0}^{N-1}\bigg(H_k(t)X_k^N(t) - H_k(0)X_k^N(0) - \int_0^t X_k^N(s)\partial_s H_k ds\bigg)\bigg]\bigg\} \\ &= \exp\Big\{ -\ell N\Big[J_H(X_t^N) + O(1/N)\Big]\Big\} , \end{aligned}$$

where

$$J_H(u) \;=\; \int_0^t \int_{\mathbb{T}} \Big[b\big(u(s,y)\big)\big(1-e^{H(s,y)}\big) + d\big(u(s,y)\big)\big(1-e^{-H(s,y)}\big)$$

$$- u(s,y)\Big(\Delta H(s,y) + \big(\nabla H(s,y)\big)^2\Big)\Big]\, dy\, ds$$
$$+ \int_{\mathbb{T}} \Big[H(t,y)u(t,y) - H(0,y)u(0,y) - \int_0^t u(s,y)\partial_s H(s,y)\, ds \Big],$$

finishing the proof. □

4.2 Large deviations upper bound

With the aid of Theorem 4.1.1, we will get the upper bound for the large deviations, since we guarantee the existence of the Radon-Nikodym derivative and therefore we can find the rate function. Recall that $\mathbb{P}_N$, $\mathbb{E}_N$ denote the probability and expectation, respectively, on trajectories of the particle system, while P_N, E_N denote probability and expectation induced by the density of particles X^N, respectively. Furthermore, the super index H on $\mathbb{P}_N^H$, $\mathbb{E}_N^H$, P_N^H, E_N^H have analogous meaning, but considering instead the perturbed process defined on Section 2.2. Let $\mathcal{O} \subseteq \mathcal{D} = \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ be an open set. Then

$$\begin{aligned} P_N\big[\mathcal{O}\big] \;&=\; \mathbb{P}_N\big[X^N \in \mathcal{O}\big] \;=\; \mathbb{E}_N\big[\mathbb{1}_{[X^N\in\mathcal{O}]}\big] \;=\; \mathbb{E}_N\Big[\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\frac{\mathbf{d}\mathbb{P}_N^H}{\mathbf{d}\mathbb{P}_N}\mathbb{1}_{[X^N\in\mathcal{O}]}\Big] \\ &=\; \mathbb{E}_N\Big[e^{-\ell N J_H(X^N)} e^{\ell N J_H(X^N)}\mathbb{1}_{[X^N\in\mathcal{O}]}\Big] \;\le\; \sup_{x\in\mathcal{O}} e^{-\ell N J_H(x)}\mathbb{E}_N\Big[e^{\ell N J_H(X^N)}\mathbb{1}_{[X^N\in\mathcal{O}]}\Big] \\ &\le\; \sup_{x\in\mathcal{O}} e^{-\ell N J_H(x)}, \end{aligned}$$

in the last inequality we use the fact that

$$\mathbb{E}_N\Big[e^{\ell N J_H(X^N)}\mathbb{1}_{[X^N\in\mathcal{O}]}\Big] \;=\; \mathbb{E}_N^H\big[\mathbb{1}_{[X^N\in\mathcal{O}]}\big] \;\le\; 1\,.$$

Therefore,

$$\limsup_{N\to\infty} \frac{1}{\ell N}\log P_N\big[\mathcal{O}\big] \;\le\; -\inf_{x\in\mathcal{O}} J_H(x)\,.$$

Optimizing over the set of perturbations, we then get

$$\limsup_{N\to\infty} \frac{1}{\ell N}\log P_N\big[\mathcal{O}\big] \;\le\; -\sup_H \inf_{x\in\mathcal{O}} J_H(x)\,. \tag{4.7}$$

To pass to compact sets, we will apply the classical *Minimax Lemma*. To be used in the sequel, we recall that

$$\limsup_{n\to\infty}\frac{1}{a_n}\log(b_n+c_n) \;=\; \max\Big\{\limsup_{n\to\infty}\frac{1}{a_n}\log b_n\,,\; \limsup_{n\to\infty}\frac{1}{a_n}\log c_n\Big\} \tag{4.8}$$

for any sequence of real numbers such that $a_n \to \infty$ and $b_n, c_n > 0$.

Proposition 4.2.1 (Minimax Lemma). *Let $\mathcal{K} \subseteq S$ compact, where (S,d) is a Polish space. Given $\{-J_H\}_H$ a family of upper semi-continuous functions, it holds that*

$$\inf_{\mathcal{O}_1,\dots,\mathcal{O}_M} \max_{1\le j\le M} \inf_H \sup_{x\in\mathcal{O}_j} -J_H(x) \;\le\; \sup_{x\in\mathcal{K}} \inf_H -J_H(x)\,, \tag{4.9}$$

where the first infimum is taken over all finite open coverings $\mathcal{O}_1,\dots,\mathcal{O}_M$ of $\mathcal{K}$.

For a proof of the Proposition above, see [12, page 363] for instance. Let now $\mathcal{K}$ be a compact set of $\mathcal{D}([0,T], C(\mathbb{T}))$. Taking $\{\mathcal{O}_1, \dots, \mathcal{O}_M\}$ a finite open covering of $\mathcal{K}$, then

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\big[\mathcal{K}\big] \;\leq\; \limsup_{N\to\infty} \frac{1}{\ell N} \log\big(P_N\big[\mathcal{O}_1\big] + \cdots + P_N\big[\mathcal{O}_M\big]\big)\,,$$

by (4.8) and (4.7) we have that

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\big[\mathcal{K}\big] \;&\leq\; \max_{1\leq j\leq M} \left\{ \limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\big[\mathcal{O}_j\big] \right\} \\
&\leq\; \max_{1\leq j\leq M} \left\{ -\sup_H \inf_{x\in\mathcal{O}_j} J_H(x) \right\} \\
&\leq\; \inf_{\substack{\mathcal{O}_1,\dots,\mathcal{O}_M \\ \text{open covering}}} \max_{1\leq j\leq M} \left\{ -\sup_H \inf_{x\in\mathcal{O}_j} J_H(x) \right\} \\
&\overset{(4.9)}{\leq}\; -\inf_{x\in\mathcal{K}} \sup_H J_H(x)\,,
\end{aligned}$$

note that the last inequality it is true because J_H is continuous functional in the Skorohod topology (see [2]). This furnishes the upper bound for compact sets. The next proposition is the usual key to pass to closed sets. Denote by $\{P_n\}_{n\in\mathbb{N}}$ a general sequence of probability measures on some metric space Ω.

Definition 4.2.2. *A sequence of measures $\{P_n\}_{n\in\mathbb{N}}$ on Ω is said to be exponentially tight if, for any $b < \infty$, there exists a compact set $\mathcal{K}_b \subseteq \Omega$ such that*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{K}_b^{\complement}\big] \;\leq\; -b\,, \tag{4.10}$$

where a_n is a constant depending on n.

Proposition 4.2.3. *Suppose that $\{P_n\}_{n\in\mathbb{N}}$ is exponentially tight and we have the large deviations upper bound for compact sets, that is, for each compact set $\mathcal{K} \subseteq \Omega$, it holds that*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{K}\big] \;\leq\; -\inf_{x\in\mathcal{K}} I(x)\,. \tag{4.11}$$

Then, for any closed $\mathcal{C} \subseteq \Omega$,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{C}\big] \;\leq\; -\inf_{x\in\mathcal{C}} I(x)\,.$$

Proof. Note that, given the subsets $\mathcal{C}$ and $\mathcal{K}_b$ of Ω, with closed $\mathcal{C}$ and compact $\mathcal{K}_b$, we have that

$$\frac{1}{a_n} \log P_n\big[\mathcal{C}\big] \;\leq\; \frac{1}{a_n} \log\Big(P_n\big[\mathcal{C}\cap\mathcal{K}_b\big] + P_n\big[\mathcal{K}_b^{\complement}\big]\Big).$$

Taking the limsup and using (4.8) in the above equation, we have that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{C}\big] \;\leq\; \max\left\{ \limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{C}\cap\mathcal{K}_b\big]\,,\; \limsup_{n\to\infty} \frac{1}{a_n} P_n\big[\mathcal{K}_b^{\complement}\big] \right\}.$$

Since $\mathcal{K}_b \cap \mathcal{C}$ is compact and by (4.11) and (4.10),

$$\limsup_{n\to\infty} \frac{1}{a_n} \log P_n\big[\mathcal{C}\big] \;\leq\; \max\left\{ -\inf_{x\in\mathcal{C}\cap\mathcal{K}_b} I(x)\,,\; -b \right\} \;\leq\; \max\left\{ -\inf_{x\in\mathcal{C}} I(x)\,,\; -b \right\}.$$

Taking $b\to\infty$, we conclude the proof. □

In view Proposition above, in order to prove the large deviations upper bound, it remains to assure exponential tightness for the sequence of probability measures P_N on $\mathcal{D}$ induced by the random element X^N and the probability $\mathbb{P}_N$. The next propositions will be necessary for this conclusion. Denote by $\|\cdot\|_1$ the L^1-norm on $\mathbb{T}$ with respect to the Lebesgue measure.

Proposition 4.2.4. *Let $C\in\mathbb{R}$ be such that $C-\|X^N(0)\|_1 > T\|b\|_\infty$. Then,*

$$\frac{1}{\ell N}\log \mathbb{P}_N\left[\sup_{t\in[0,T]} \|X^N(t)\|_1 > C\right] \;\leq\; -I\big(C-\|X^N(0)\|_1\big)\,, \tag{4.12}$$

for any $N\in\mathbb{N}$, where $I(x)=x\log\big(\frac{x}{\|b\|_\infty}\big)-x+\|b\|_\infty$.

Proof. First of all, we note that $I(x)$ is the rate function for sums of i.i.d random variables with Poisson distribution of parameter $\|b\|_\infty$. To prove (4.12), we consider a birth process $W^N(t)$ on the state space $\mathbb{N}$ which jump rate from k to $k+1$ is given by $N\ell\|b\|_\infty$ for any $k\in\mathbb{N}$ and $W^N(0)=\sum_{k\in\mathbb{T}_N}\eta_k(0)$. Recall that, by assumption, the initial quantity of particles is a deterministic value. Since the rate at which a particle is created somewhere in the particle system $\eta(t)$ is smaller than $N\ell\|b\|_\infty$, it is a standard procedure to construct a coupling between $W^N(t)$ and $\eta(t)$ such that, almost surely,

$$W^N(t) \;\geq\; \sum_{k\in\mathbb{T}_N}\eta_k\,,\quad \forall\, t\in[0,T]\,,$$

which implies that, almost surely,

$$\frac{1}{\ell N}W^N(t) \;\geq\; \frac{1}{\ell N}\sum_{k\in\mathbb{T}_N}\eta_k \;=\; \|X^N(t)\|_1\,,\quad \forall\, t\in[0,T]\,, \tag{4.13}$$

Abusing of notation, denote the coupling between $\eta(t)$ and $W^N(t)$ also by $\mathbb{P}_N$, and by $\widetilde{P}$ the marginal probability concerning $W^N(t)$. Therefore, in view of (4.13),

$$\begin{aligned}\mathbb{P}_N\left[\sup_{t\in[0,T]}\|X^N(t)\|_1 > C\right] \;&\leq\; \mathbb{P}_N\left[\sup_{t\in[0,T]}\frac{1}{\ell N}W^N(T) > C\right]\\ &\leq\; \widetilde{P}\Big[W^N(T)-W^N(0) > \ell N C - W^N(0)\Big]\,. \end{aligned}\tag{4.14}$$

Since the distribution of $W^N(T) - W^N(0)$ is Poisson of parameter $\ell NT\|b\|_\infty$, and sum of independent Poisson random variables is Poisson, the probability in (4.14) is equal to

$$\widetilde{P}\bigg(\frac{Z_1+\cdots+Z_{\ell N}}{\ell N} > C - \frac{W^N(0)}{\ell N}\bigg) \;=\; \widetilde{P}\bigg(\frac{Z_1+\cdots+Z_{\ell N}}{\ell N} > C - \|X^N(0)\|_1\bigg)\,,$$

where $Z_1, Z_2, \dots$ are i.i.d. random variables of distribution Poisson$\big(T\|b\|_\infty\big)$ on some probability space with probability $\widetilde{P}$. Since $C - \|X^N(0)\|_1 > T\|b\|_\infty$, standard large deviations for sums of i.i.d. random variables gives us that

$$\frac{1}{\ell N}\log \widetilde{P}\bigg(\frac{Z_1+\cdots+Z_{\ell N}}{\ell N} > C - \|X^N(0)\|_1\bigg) \;\leq\; -I\Big(C - \|X^N(0)\|_1\Big)\,,$$

where $I(x) = x\log\big(\frac{x}{\|b\|_\infty}\big) - x + \|b\|_\infty$, concluding the proof. □

Proposition 4.2.5. *For every continuous function $H : [0,+\infty)\times\mathbb{T}\to\mathbb{R}$ and $\varepsilon>0$,*

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\bigg[\sup_{|t-s|<\delta}\Big|\big\langle X^N(t),H(t)\big\rangle - \big\langle X^N(s),H(s)\big\rangle\Big| > \varepsilon\bigg] \;=\; -\infty\,. \tag{4.15}$$

Proof. Partitioning the time interval $[0,T]$ in intervals of size at most δ and applying the triangular inequality together with (4.8), one can see that it is enough to assure that

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\bigg[\sup_{k\delta\leq t\leq (k+1)\delta}\Big|\big\langle X^N(t),H(t)\big\rangle - \big\langle X^N(k\delta),H(k\delta)\big\rangle\Big| > \varepsilon\bigg] \;=\; -\infty \tag{4.16}$$

in order to have (4.15). Therefore, our goal from now on is to prove (4.16) for fixed $K\in\{1,\dots,\lfloor T/\delta\rfloor\}$. Since $|x| = \max\{x,-x\}$ and using (4.8), it is enough to show that

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\bigg[\sup_{K\delta\leq t\leq (K+1)\delta}\Big(\big\langle X^N(t),H(t)\big\rangle - \big\langle X^N(K\delta),H(K\delta)\big\rangle\Big) > \varepsilon\bigg] = -\infty \tag{4.17}$$

and

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\bigg[\sup_{K\delta\leq t\leq (K+1)\delta}\Big(\big\langle X^N(t),H(t)\big\rangle - \big\langle X^N(K\delta),H(K\delta)\big\rangle\Big) < -\varepsilon\bigg] = -\infty\,. \tag{4.18}$$

We will only prove (4.17) whereas the argument for (4.18) is similar. Analogously to (4.1), we may find

$$A_a^N(t) \;=\; \int_{K\delta}^t \frac{1}{N}\sum_{k=0}^{N-1}\Big[b\big(X_k^N(s)\big)\big(1-e^{aH_k}\big) + d\big(X_k^N(s)\big)\big(1-e^{-aH_k}\big)$$

$$- X_k^N(s)\Big(a\Delta_N H_k + \frac{a^2}{2}\Big(\big(\nabla_N^+ H_k\big)^2 + \big(\nabla_N^- H_k\big)^2\Big) + O(1/N)\Big)\Big] ds$$
$$+ \frac{a}{N}\sum_{k=0}^{N-1}\Big(H_k(t)X_k^N(t) - H_k(K\delta)X_k^N(K\delta) - \int_{K\delta}^t X_k^N(s)\partial_s H_k ds\Big)$$

such that $\exp\Big\{-\ell N A_a^N\Big\}$ is a mean-one martingale. Define R_a^N by the equality

$$\begin{aligned}
R_a^N(t) \;&=\; A_a^N(t) - \frac{a}{N}\sum_{k=0}^{N-1}\Big(H_k(t)X_k^N(t) - H_k(K\delta)X_k^N(K\delta)\Big)\\
&=\; A_a^N(t) - a\Big[\big\langle X^N(t), H(t)\big\rangle - \big\langle X^N(K\delta), H(K\delta)\big\rangle\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathbb{P}_N\Bigg[\sup_{K\delta\le t\le (K+1)\delta}\Big(\big\langle X^N(t), H(t)\big\rangle - \big\langle X^N(K\delta), H(K\delta)\big\rangle\Big) > \varepsilon\Bigg]\\
&\quad= \mathbb{P}_N\Bigg[\sup_{K\delta\le t\le (K+1)\delta}\big(A_a^N(t) - R_a^N(t)\big) > a\varepsilon\Bigg]\\
&\quad= \mathbb{P}_N\Bigg[\sup_{K\delta\le t\le (K+1)\delta} e^{\ell N\big(A_a^N(t)-R_a^N(t)\big)} > e^{a\varepsilon\ell N}\Bigg].
\end{aligned}$$

Define the event

$$E \;=\; \Big[\sup_{t\in[0,T]}\|X^N(t)\|_1 \le C\Big].$$

Restrict to E, it is straightforward to check that

$$|R_a^N| \le m(H,b,d)C\delta\,,$$

where $m(H,b,d)$ is a constant depending only on H, on its first and second derivatives and on the Lipschitz constant of b and d. Note that the factor δ appears since the integral in time is taken over the interval $[K\delta, t]$. Hence, partitioning into E and $E^{\complement}$, we have that

$$\begin{aligned}
&\mathbb{P}_N\Bigg[\sup_{K\delta\le t\le (K+1)\delta} e^{\ell N(A_a^N(t)-R_a^N(t))} > e^{a\varepsilon\ell N}\Bigg] \qquad (4.19)\\
&\le\; \mathbb{P}_N\Bigg[\sup_{K\delta\le t\le (K+1)\delta} e^{\ell N A_a^N(t)} > e^{\ell N(a\varepsilon - m(H,b,d)C\delta)}\Bigg] + \mathbb{P}_N\big[E^{\complement}\big].
\end{aligned}$$

By Doob's inequality, the right-hand side of above is bounded from above by

$$\frac{\mathbb{E}_N\big[e^{\ell N A_a^N(t)}\big]}{e^{\ell N(a\varepsilon - m(H,b,d)C\delta)}} + \mathbb{P}_N\big[E^{\complement}\big] \;=\; \exp\{-\ell N(a\varepsilon - m(H,b,d)C\delta)\} + \mathbb{P}_N\big[E^{\complement}\big].$$

Applying the logarithm function in (4.19), dividing it by ℓN, taking the $\limsup_N$ and recalling (4.8) give us that

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\Bigg[\sup_{K\delta\le t\le(K+1)\delta} e^{\ell N(A_a^N(t)-R_a^N(t))}>e^{a\varepsilon\ell N}\Bigg]\\ &\le\ \max\Big\{-(a\varepsilon-m(H,b,d)C\delta)\ ,\ \limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\big[E^{\complement}\big]\Big\}\,.\end{aligned}$$

Applying Proposition 4.2.4, we can bound the expression above by

$$\begin{aligned}&\max\Big\{-a\varepsilon+m(H,b,d)C\delta\ ,\ \limsup_{N\to\infty}-I\big(C-\|X^N(0)\|_1\big)\Big\}\\ &=\ \max\Big\{-a\varepsilon+m(H,b,d)C\delta\ ,\ -I\big(C-\|\psi(0)\|_1\big)\Big\}\,.\end{aligned}$$

Since $\lim_{x\to\infty} I(x)=\infty$, we are allowed to first choose C large, then δ small, and then finally a large, leading us to conclude that

$$\limsup_{N\to\infty}\frac{1}{\ell N}\log\mathbb{P}_N\Bigg[\sup_{K\delta\le t\le(K+1)\delta} e^{\ell N(A_a^N(t)-R_a^N(t))}>e^{a\varepsilon\ell N}\Bigg]\ =\ -\infty\,,$$

finishing the proof. □

Proposition 4.2.6. *The sequence of measures $\{P_N\}_{N\in\mathbb{N}}$ on $\mathcal{D}$ is exponentially tight.*

Proof. Using (4.15), we obtain the sequence of compact sets satisfying (4.10). Define the sets

$$\begin{aligned}L_c\ &=\ \big\{u\in\mathcal{D}:\|u_0\|_\infty\le c\big\}\\ C_{\delta,1/n}\ &=\ \Big\{u\in\mathcal{D}:\sup_{|t-s|<\delta}\|u_t-u_s\|_\infty\le 1/n\Big\}\,,\\ A\ &=\ \big(\cap_{n=1}^\infty C_{\delta,1/n}\big)\cap L_c\,.\end{aligned}$$

By Arzelá-Ascoli's Theorem, the set A is pre-compact, hence $\overline{A}$ is compact. Taking $\{H_j\}_{j\in\mathbb{N}}$ a dense set in $C(\mathbb{T})$, let us define

$$C_{\delta,1/n}^{H_j}\ =\ \Big\{u\in\mathcal{D}:\sup_{|t-s|<\delta}\Big|\int u_t(x)H_j(t,x)dx-\int u_s(x)H_j(s,x)dx\Big|\le 1/n\Big\}$$

and

$$B_\delta\ =\ L_c\cap\big(\cap_{j,n=1}^\infty C_{\delta,1/n}^{H_j}\big)\,.$$

Our goal is to prove that $\overline{B_\delta}$ is compact, so it suffices to verify that $B_\delta\subseteq A$. Let $u\in\big(\cap_{n=1}^\infty C_{\delta,1/n}\big)^{\complement}$, then there exists $n_0\in\mathbb{N}$ such that $u\in C_{\delta,1/n_0}^{\complement}$, that is, there exists $|t-s|<\delta$ such that $\|u_t-u_s\|_\infty>1/n$. Since $\{H_j\}_j$ is dense, there exists H_{j_0} with

$\big|\int u_t(x)H_{j_0}(x,t)dx - \int u_s(x)H_{j_0}(x,s)dx\big| > 1/n$, hence $u \in \big(C^{H_{j_0}}_{\delta,1/n}\big)^{\complement}$. Finally we show (4.10). Note that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[\overline{B}_\delta^{\complement}\right] \\
&= \limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[\left(\overline{L_c \cap \left(\cap_{j,n=1}^{\infty} C^{H_j}_{\delta,1/n}\right)}\right)^{\complement}\right] \\
&\leq \limsup_{N\to\infty} \frac{1}{\ell N} \log \left[P_N\left[L_c^{\complement}\right] + \sum_{j,n=1}^{\infty} P_N\left[\left(C^{H_j}_{\delta,1/n}\right)^{\complement}\right]\right] \\
&\leq \max\left\{\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[L_c^{\complement}\right],\ \limsup_{N\to\infty} \frac{1}{\ell N} \log\left[\sum_{j,n=1}^{\infty} P_N\left[\left(C^{H_j}_{\delta,1/n}\right)^{\complement}\right]\right]\right\},
\end{aligned}$$

where in the second inequality we used (4.8). Since

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log \mathbb{P}_N\left[\|X^N(0)\|_\infty > c\right] = -\infty ,$$

then

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[\overline{B}_\delta^{\complement}\right] \leq \limsup_{N\to\infty} \frac{1}{\ell N} \log\left[\sum_{j,n=1}^{\infty} P_N\left[\left(C^{H_j}_{\delta,1/n}\right)^{\complement}\right]\right], \tag{4.20}$$

By (4.15), there exists δ_0 such that

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[\left(C^{H_j}_{\delta_0,\varepsilon}\right)^{\complement}\right] \leq \frac{-b_{\delta_0}}{\varepsilon} ,$$

and there exists N_0 such that for all $N > N_0$,

$$\frac{1}{\ell N} \log P_N\left[\left(C^{H_j}_{\delta_0,\varepsilon}\right)^{\complement}\right] \leq \frac{-b_{\delta_0}}{\varepsilon} .$$

Therefore,

$$\sum_{j,n=1}^{\infty} P_N\left[\left(C^{H_j}_{\delta,1/n}\right)^{\complement}\right] \leq \sum_{j,n=1}^{\infty} \exp\{-b_\delta \ell N n\} = \frac{e^{-b_\delta \ell N}}{1-e^{-b_\delta \ell N}} \leq 2e^{-b_\delta \ell N}.$$

Then, coming back to (4.20),

$$\limsup_{N\to\infty} \frac{1}{\ell N} \log P_N\left[\overline{B}_\delta^{\complement}\right] < \limsup_{N\to\infty} \frac{1}{\ell N} \log\left(2e^{-b_\delta \ell N}\right) = -b_\delta .$$

Now, taking $b = b_\delta$ we obtain the exponential tightness (4.10) hence finishing the proof. □

Therefore, with Lemma 4.2.3 and Proposition 4.2.6 at hand we have concluded the proof of the upper bound for large deviations.

4.3 Large deviations lower bound for smooth profiles

In this section we will prove the large deviations principle lower bound considering profiles in $\mathcal{D}^{\alpha}_{\text{pert}}$, defined in Chapter 2. First, we obtain a non-variational formulation of the rate functional I for profiles ψ which are solutions of the partial differential equation corresponding to the perturbed process for some perturbation H.

Proposition 4.3.1. *Given $H \in C^{1,2}$, let $\psi = \psi^H$ be the unique solution of* (2.5). *Then,*

$$\boldsymbol{I}(\psi) \overset{def}{=} \sup_G J_G(\psi) = J_H(\psi) = \int_0^t \int_{\mathbb{T}} \Big[(\partial_x H)^2 \psi + b(\psi)\,\boldsymbol{\Gamma}(H) + d(\psi)\,\boldsymbol{\Gamma}(-H)\Big]\,dx\,ds\,, \tag{4.21}$$

where $\boldsymbol{\Gamma}(y) = 1 - e^y + y\,e^y$, $y \in \mathbb{R}$.

Proof. Multiplying the PDE (2.5) by a test function $G \in C^{1,2}$ and integrating in space and time, we get that

$$\int_{\mathbb{T}} \int_0^t G\partial_t \psi\,ds\,dx = \int_{\mathbb{T}} \int_0^t G\partial_x^2 \psi - 2G\partial_x\big(\psi \partial_x H\big) + G\big[e^H b(\psi) - e^{-H} d(\psi)\big]\,ds\,dx\,.$$

Using integration by parts and that

$$\begin{aligned} Ge^H b(\psi) &= b(\psi)\overline{\Gamma}(G,H) - b(\psi)(1-e^G)\,,\\ -Ge^{-H} d(\psi) &= d(\psi)\overline{\Gamma}(-G,-H) - d(\psi)(1-e^{-G})\,, \end{aligned}$$

where $\overline{\Gamma}(x,y) = 1 - e^x + x\,e^y$, we conclude that

$$\begin{aligned} &\int_{\mathbb{T}} \Big[G(t,x)\psi(t,x) - G(0,x)\psi(0,x) - \int_0^t \psi(s,x)\partial_t G(s,x)\,ds\Big]\,dx\\ &\quad = \int_0^t \int_{\mathbb{T}} \partial_x^2 G(s,x)\psi(s,x)\,dx\,ds + \int_0^t \int_{\mathbb{T}} 2\psi(s,x)\partial_x G(s,x)\partial_x H(s,x)\,dx\,ds\\ &\qquad + \int_0^t \int_{\mathbb{T}} b(\psi(s,x))\overline{\Gamma}\big(G(s,x),H(s,x)\big) - b(\psi(s,x))\big(1-e^{G(s,x)}\big)\\ &\qquad + d(\psi(s,x))\overline{\Gamma}\big(-G(s,x),-H(s,x)\big) - d(\psi(s,x))\big(1-e^{-G(s,x)}\big)dx\,ds\,, \end{aligned}$$

Recall the definition of J_H in (2.8). The equality above allows us to deduce that

$$J_G(\psi) = \int_0^t \int_{\mathbb{T}} \Big[-\psi(\partial_x G)^2 + 2\psi\partial_x G\partial_x H + b(\psi)\overline{\Gamma}(G,H) + d(\psi)\overline{\Gamma}(-G,-H)\Big]dx\,ds\,.$$

Finally, noting that $2\partial_x G\partial_x H = -\big(\partial_x G - \partial_x H\big)^2 + (\partial_x G)^2 + (\partial_x H)^2$, we arrive at

$$J_G(\psi) = \int_0^t \int_{\mathbb{T}} \Big[-\big(\partial_x G - \partial_x H\big)^2\psi + (\partial_x H)^2\psi + b(\psi)\overline{\Gamma}(G,H) + d(\psi)\overline{\Gamma}(-G,-H)\Big]\,dx\,ds\,.$$

Fix $y \in \mathbb{R}$. Since the function $x \mapsto \overline{\Gamma}(x,y)$ assumes its maximum at $x = y$ and $-(\partial_x G - \partial_x H)^2$ assumes its maximum at $G = H$, we conclude that

$$\boldsymbol{I}(\psi) = \sup_G J_G(\psi) = J_H(\psi).$$

Since $\boldsymbol{\Gamma}(y) = \overline{\Gamma}(y,y)$, we obtain (4.21). □

Solutions of (2.5) for some H provide the special representation above for the rate function. It is thus natural to find the set of profiles ψ for which we may find a perturbation H fulfilling the requirements in order to permit the high density limit (towards ψ).

Proposition 4.3.2. *Let $\psi \in C^{2,3}$ such that $\psi \geq \varepsilon$ for some $\varepsilon > 0$. Then, there exists a unique solution $H \in C^{1,2}$ of the elliptic equation*

$$\partial_x^2 H \; + \; \frac{\partial_x \psi}{\psi} \partial_x H \; = \; \frac{\partial_x^2 \psi - \partial_t \psi}{2\psi} + e^H b(\psi) - e^{-H} d(\psi)\,. \tag{4.22}$$

Proof. For each fixed time $t \in [0,T]$, equation (4.22) can seen a non-linear second order ordinary differential equation on the interval $[0,1]$. As an ODE in $[0,1]$ any of its solutions can be written as the sum of a particular solution of (4.22) plus some solution of the homogeneous part

$$\partial_x^2 H \; + \; \frac{\partial_x \psi}{\psi} \partial_x H \; = \; 0\,. \tag{4.23}$$

Solving (4.23) and then properly choosing constants, allow to find a particular solution of (4.22) such that $H(0) = H(1)$, $\partial_x H(0) = \partial_x H(1)$ and $\partial_x^2 H(0) = \partial_x^2 H(1)$, that is, such a solution H belongs to $C^{1,2}$. Details are omitted here. □

By Proposition 4.3.1, a profile which is a solution of (2.5) for some H provides a special representation for the rate function. This together with Proposition 4.3.2 motivates the definition of $\mathcal{D}^{\alpha}_{\text{pert}}$ given in Section 2.1.

Due to Proposition 4.3.2 and Remark 2.2.3, given $\psi \in \mathcal{D}^{\alpha}_{\text{pert}}$, we can find $H = H(\psi) \in C^{1,2}$ such that the assumptions of Theorem 2.2.1 are satisfied. In words, the perturbed process (under the perturbation H) has a high density limit, and the limiting profile is the aforementioned ψ. We are now in position to prove the lower bound for trajectories in $\mathcal{D}^{\alpha}_{\text{pert}}$. Before, we need to gather some ingredients, which will be given by the next four lemmas.

Lemma 4.3.3. *Let $C \in \mathbb{R}$ be such that $C - \|X^N(0)\|_1 > T\|be^H\|_\infty$. Then,*

$$\frac{1}{\ell N} \log \mathbb{P}_N^H \Big[\sup_{t\in[0,T]} \|X^N(t)\|_1 > C \Big] \; \leq \; -I\big(C - \|X^N(0)\|_1\big)\,, \tag{4.24}$$

for any $N \in \mathbb{N}$, where $I(x) = x \log\big(\frac{x}{\|be^H\|_\infty}\big) - x + \|be^H\|_\infty$.

Proof. Note that the probability above is the one associated to the *perturbed* process. The proof of the inequality (4.24) is exactly the same as that one of Proposition 4.2.4 once we replace $\|b\|_\infty$ by $\|be^H\|_\infty$. □

Lemma 4.3.4. *The expectation* $\mathbb{E}_N^H\big[\,\big|\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\big|^2\,\big]$ *is uniformly bounded in* $N\in\mathbb{N}$.

Proof. By Proposition 4.1.1, it not difficult to see that

$$\left|\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\right| \;\leq\; f(X^N) \;\stackrel{\text{def}}{=}\; \bar{c}\int_{\mathbb{T}}\Big(|X^N(t)|+|X^N(0)|+\int_0^t|X^N(s)|\,ds\Big)\,dx \tag{4.25}$$

for some $\bar{c}=\bar{c}(H)>0$. Observe that

$$f(X^N)\;\leq\;\bar{c}\cdot(2+t)\sup_{t\in[0,T]}\|X^N(t)\|_1\,.$$

As a consequence of Lemma 4.3.3,

$$\begin{aligned}\frac{1}{\ell N}\log\mathbb{P}_N^H\bigg[\frac{f(X^N)}{\bar{c}(2+t)}>C\bigg] \;&\leq\; \frac{1}{\ell N}\log\mathbb{P}_N^H\bigg[\sup_{t\in[0,T]}\|X^N(t)\|_1>C\bigg]\\ &\leq\; -I\big(C-\|X^N(0)\|_1\big)\,,\end{aligned}$$

for any $N\in\mathbb{N}$, where C and I above are the same as in the statement of Lemma 4.3.3. Replacing C by $\sqrt{k}/\bar{c}(2+t)$, where $k\in\mathbb{N}$ is large enough, we conclude that

$$\mathbb{P}_N^H\Big[f(X^N)>\sqrt{k}\Big]\;\leq\;\exp\bigg\{-\ell N\,I\bigg(\frac{\sqrt{k}}{\bar{c}(2+t)}-\|X^N(0)\|_1\bigg)\bigg\}\,,$$

thus

$$\begin{aligned}\mathbb{P}_N^H\Big[f(X^N)^2>k\Big]\;&\leq\;\exp\bigg\{-\ell N\,I\bigg(\frac{\sqrt{k}}{\bar{c}(2+t)}-\|X^N(0)\|_1\bigg)\bigg\}\\ &\leq\;\exp\bigg\{-I\bigg(\frac{\sqrt{k}}{\bar{c}(2+t)}-\|X^N(0)\|_1\bigg)\bigg\}\,,\end{aligned}$$

for all $k\geq k_0$ with $k_0\in\mathbb{N}$. Keep in mind that the choice of k_0 does not depend on ℓ neither N, see the statement of Lemma 4.3.3. Since $I(x)=x\log\big(\frac{x}{\|be^H\|_\infty}\big)-x+\|be^H\|_\infty$, some simple analysis permits to deduce that

$$\sum_{k\geq k_0}\mathbb{P}_N^H\Big[f(X^N)^2>k\Big]\;\leq\;c_1\;<\;\infty\,,$$

for some suitably large $k_0\in\mathbb{N}$. This allows to finish the proof. □

Lemma 4.3.5. *Let* $\psi\in\mathcal{D}^\alpha_{\text{pert}}$*,* $\mathcal{O}$ *be an open set of* $\mathcal{D}$ *such that* $\psi\in\mathcal{O}$ *and* $H\in C^{1,2}$ *the solution of* (4.22)*. Then*

$$\lim_{N\to\infty}\mathbb{E}_N^H\bigg[\mathbb{1}_{[X^N\in\mathcal{O}^\complement]}\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg]\;=\;0\,. \tag{4.26}$$

Proof. By Lemma (4.3.4) and Cauchy-Schwarz inequality,

$$\mathbb{E}_N^H\bigg[\mathbb{1}_{[X^N\in\mathcal{O}^\complement]}\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg]\;\leq\;\sqrt{\mathbb{P}_N^H[X^N\in\mathcal{O}^\complement]}\sqrt{\mathbb{E}_N^H\Big[\Big(\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\Big)^2\Big]}\,,$$

which proves (4.26) due to Theorem 2.2.1, concluding the proof. □

We make now the classical connection between the rate function and the entropy between the process of reference and the perturbed process.

Lemma 4.3.6. *Let*

$$\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;\stackrel{def}{=}\; \mathbb{E}_N^H\bigg[\log\frac{\mathbf{d}\mathbb{P}_N^H}{\mathbf{d}\mathbb{P}_N}\bigg] \tag{4.27}$$

be the relative entropy of $\mathbb{P}_N^H$ *with respect to* $\mathbb{P}_N$. *Then,*

$$\lim_{N\to\infty}\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;=\; \boldsymbol{I}(\psi)\,,$$

where ψ *is the (unique) solution of* (2.5).

Proof. Note that

$$\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;=\; \frac{1}{\ell N}\mathbb{E}_N^H\bigg[\log\frac{\mathbf{d}\mathbb{P}_N^H}{\mathbf{d}\mathbb{P}_N}\bigg] \;=\; -\frac{1}{\ell N}\mathbb{E}_N^H\bigg[\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg]$$

Recalling the expression 4.1 for the Radon-Nikodym derivative, we get that

$$\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;=\; \mathbb{E}_N^H\Big[J_H(X^N)+O(1/N)\Big]\,.$$

By Lemma 4.3.4, $\{J_H(X^N)\}$ is a uniformly integrable sequence (with respect to $\mathbb{P}_N^H$). Since $J_H:\mathcal{D}\to\mathbb{R}$ is a continuous function and $\mathbb{P}_N^H$ converges weakly to a delta of Dirac at ψ, we conclude that

$$\lim_{N\to\infty}\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;=\; J_H(\psi) \;=\; \boldsymbol{I}(\psi)\,,$$

by Proposition 4.3.1, which finishes the proof. □

We are in position to finally prove Proposition 2.12.

Proof of lower bound for profiles in $\mathcal{D}^{\alpha}_{\text{pert}}$. Fix an open set $\mathcal{O}$. Given $\psi\in\mathcal{O}\cap\mathcal{D}^{\alpha}_{\text{pert}}$, there exists $H\in C^{1,2}$ such that ψ is solution of (2.5) and $\|\partial_x H\|_\infty<\pi\sqrt{\alpha}$. Denote by $\mathbb{P}_N^{H,\mathcal{O}}$ the probability on the space $\mathcal{D}_{\Omega_N}$ given by

$$\mathbb{P}_N^{H,\mathcal{O}}[A] \;\stackrel{\text{def}}{=}\; \frac{\mathbb{P}_N^H[A, X^N\in\mathcal{O}]}{\mathbb{P}_N^H[X^N\in\mathcal{O}]}\,,$$

for any A measurable subset of $\mathcal{D}_{\Omega_N}$. Under this definition,

$$\begin{aligned}\frac{1}{\ell N}\log P_N[\mathcal{O}] \;&=\; \frac{1}{\ell N}\log\mathbb{P}_N[X^N\in\mathcal{O}]\\ &=\; \frac{1}{\ell N}\log\mathbb{E}_N\bigg[\mathbb{1}_{[X^N\in\mathcal{O}]}\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\frac{\mathbf{d}\mathbb{P}_N^H}{\mathbf{d}\mathbb{P}_N}\bigg]\\ &=\; \frac{1}{\ell N}\log\mathbb{E}_N^H\bigg[\mathbb{1}_{[X^N\in\mathcal{O}]}\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\bigg]\end{aligned}$$

$$= \frac{1}{\ell N}\log \mathbb{E}_N^{H,\mathcal{O}}\left[\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\right] + \frac{1}{\ell N}\log \mathbb{P}_N^H[X^N \in \mathcal{O}]\,. \tag{4.28}$$

Since $\mathcal{O}$ is a open set and $\psi \in \mathcal{O}$, by Theorem 2.2.1 and Portmanteau's Theorem,

$$\liminf \mathbb{P}_N^H[X^N \in \mathcal{O}] \;\geq\; 1\,,$$

hence the second parcel on (4.28) converges to zero as $N \to \infty$. Since the logarithm is a concave function, by Jensen inequality the first parcel in (4.28) is bounded from below by

$$\mathbb{E}_N^{H,\mathcal{O}}\left[\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\right] \;=\; \frac{\mathbb{E}_N^H\left[\mathbb{1}_{[X^N\in\mathcal{O}]}\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\right]}{\mathbb{P}_N^H[X^N \in \mathcal{O}]}\,. \tag{4.29}$$

Adding and subtracting terms, we can rewrite (4.29) as

$$\frac{1}{\mathbb{P}_N^H\big[X^N \in \mathcal{O}\big]}\left\{-\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) - \mathbb{E}_N^H\left[\mathbb{1}_{[X^N\in\mathcal{O}^\complement]}\frac{1}{\ell N}\log\frac{\mathbf{d}\mathbb{P}_N}{\mathbf{d}\mathbb{P}_N^H}\right]\right\}, \tag{4.30}$$

Again by Theorem 2.2.1 and the Portmanteau Theorem, we have that $\mathbb{P}_N^H\big[X^N \in \mathcal{O}\big]$ goes to one as N increases to infinity. By Lemma 4.3.5 the second term inside braces in (4.30) vanishes as $N \to \infty$. Thus

$$\liminf_{N\to\infty}\frac{1}{\ell N}\log \mathbb{P}_N[\mathcal{O}] \;\geq\; \lim_{N\to\infty} -\frac{1}{\ell N}\boldsymbol{H}\big(\mathbb{P}_N^H|\mathbb{P}_N\big) \;=\; -\boldsymbol{I}(\psi)\,,$$

where the last equality has been assured in Lemma 4.3.6. Optimizing the inequality above over $\psi \in \mathcal{D}_{\text{pert}}^{\alpha}$ leads us to (2.12) hence concluding the proof. □

4.4 Large deviations lower bound for ℓ in the exponential case

In this section we will assume that $\ell(N) = e^{cN}$ in order to obtain a full large deviations principle. The scheme of proof here follows the same ideas of [11] and it is included here for sake of completeness.

Definition 4.4.1. *Denote by* $\mathcal{D}_{\text{pert}}^{\infty} \subseteq \mathcal{D} = \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ *the set of all profiles* $\psi : [0,T]\times\mathbb{T} \to \mathbb{R}$ *satisfying:*

- $\psi \in C^{2,3}$,
- $\psi \geq \varepsilon$ *for some* $\varepsilon > 0$.

Repeating *ipsis litteris* the arguments of the previous subsection, under the hypothesis that $\ell(N) = e^{cN}$ we get that, given an open set $\mathcal{O} \subset \mathcal{D}\big([0,T], C(\mathbb{T})\big)$, for any $\psi \in \mathcal{D}_{\text{pert}}^{\infty} \cap \mathcal{O}$, we have that

$$\liminf_{N\to\infty}\frac{1}{\ell N}\log \mathbb{P}_N[\mathcal{O}] \;\geq\; -\boldsymbol{I}(\psi)\,. \tag{4.31}$$

In what follows, we will say that a sequence $\rho_n \in \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ approximates $\rho_0 \in \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ if ρ_n converges to ρ_0 in the topology of $\mathcal{D}\big([0,T], C(\mathbb{T})\big)$ and

$$\lim_{n\to\infty} \boldsymbol{I}(\rho_n) \;=\; \boldsymbol{I}(\rho_0)\,. \tag{4.32}$$

To conclude the proof of the lower bound large deviations it only remains to prove that any profile $\rho_0 \in \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ such that $\boldsymbol{I}(\rho_0) < \infty$ can be approximated by a sequence $\rho_n \in \mathcal{D}^{\infty}_{\text{pert}}$. In the usual terminology, we have to assure that the set $\mathcal{D}^{\infty}_{\text{pert}}$ is $\boldsymbol{I}$*-dense*. In plain words, (4.32) together with the $\boldsymbol{I}$-density of $\mathcal{D}^{\infty}_{\text{pert}}$ imply the lower bound in Theorem 2.3.3.

Let us start by splitting the functional J_H into the H-dependent part, denoted by J^1_H, and the part which does depend on H, denoted by J^2. That is:

$$\begin{aligned} J^1_H(\rho) \;=\; & \int_{\mathbb{T}} \Big[H(t,x)\rho(t,x) - H(0,x)\rho(0,x) \Big]\, dx \\ & + \int_0^t \int_{\mathbb{T}} \Big[-\rho(s,x)\Big(\partial_s H(s,x) + \Delta H(s,x) + \big(\nabla H(s,x)\big)^2\Big) \\ & \qquad - b\big(\rho(s,x)\big)e^{H(s,x)} - d\big(\rho(s,x)\big)e^{-H(s,x)} \Big]\, dx\, ds\,, \end{aligned} \tag{4.33}$$

and

$$J^2(\rho) \;=\; \int_0^t \int_{\mathbb{T}} b\big(\rho(s,x)\big) + d\big(\rho(s,x)\big)\, dx\, ds\,. \tag{4.34}$$

Hence we define $\boldsymbol{I}^1(\rho) = \sup_{H\in C^{1,2}} J^1_H(\rho)$ if $u(\cdot,0) = \gamma(\cdot)$, and $\boldsymbol{I}^1(\rho) = \infty$ otherwise, which gives us that

$$\boldsymbol{I}(\rho) \;=\; \boldsymbol{I}^1(\rho) + J^2(\rho)\,.$$

Proposition 4.4.2. *The functional* $\boldsymbol{I}^1 : \mathcal{D}\big([0,T], C(\mathbb{T})\big) \to \mathbb{R}_+ \cup \{+\infty\}$ *is convex.*

Proof. The functions b and d are assumed to be concave, thus J^1_H is a convex function, see (4.33). Since the supremum of convex functions is a convex function, then $\boldsymbol{I}^1$ is a convex function. □

Proposition 4.4.3. *The rate function* $\boldsymbol{I} : \mathcal{D}\big([0,T], C(\mathbb{T})\big) \to \mathbb{R}_+ \cup \{+\infty\}$ *is a lower semi-continuous (l.s.c) function, that is,*

$$\liminf_{\rho\to\rho_0} \boldsymbol{I}(\rho) \;\geq\; \boldsymbol{I}(\rho_0)$$

for any $\rho_0 \in \mathcal{D}\big([0,T], C(\mathbb{T})\big)$*. Moreover,* $\boldsymbol{I}^1 : \mathcal{D}\big([0,T], C(\mathbb{T})\big) \to \mathbb{R}_+ \cup \{+\infty\}$ *is also lower semi-continuous and* J *is continuous.*

Proof. We start by noting that $J^1_H, J^2 : \mathcal{D}\big([0,T], C(\mathbb{T})\big) \to \mathbb{R}$ are continuous functionals in the Skorohod topology (see [2]) hence they are l.s.c. Since the supremum of l.s.c functions is a l.s.c function, we deduce that $\boldsymbol{I}^1$ is l.s.c. And since the sum of l.s.c functions is a l.s.c function, we conclude that $\boldsymbol{I} : \mathcal{D}\big([0,T], C(\mathbb{T})\big) \to \mathbb{R}_+ \cup \{+\infty\}$ is also a l.s.c function. □

The next proposition tell us that time discontinuous space-time profiles play no role in the large deviations behavior.

Proposition 4.4.4. *If $\rho \in \mathcal{D}\big([0,T], C(\mathbb{T})\big)$ and $\rho \notin C\big([0,T]\times\mathbb{T}\big)$, then $\boldsymbol{I}(\rho) = +\infty$.*

Proof. We claim first that, if $f:[0,T]\to\mathbb{R}$ is discontinuous at $a\in[0,T]$ and has side limits at a, and $F, G:\mathbb{R}\to\mathbb{R}$ are continuous functions, then

$$\sup_{H\in C^1([0,T])}\left\{\int_0^T f(s)\partial_s H(s)\,ds - \int_0^T F(f(s))G(H(s))\,ds\right\} \;=\; \infty\,. \tag{4.35}$$

In fact, let $H_n:[0,T]\to\mathbb{R}$ such that H_n has support in the interval $[a-1/n^2, a+1/n^2]$, $H_n\in C^\infty([0,T])$, $H_n(a)=n$ and $0\leq H_n\leq n$, that is, H_n is close to a delta of Dirac times the constant $1/n$ in the sense of Schwartz distributions.

Since the L^1-norm of H_n is of order $1/n$, it is easy to check that

$$\int_0^T F(f(s))G(H_n(s))\,ds$$

converges as $n\to\infty$. On the other hand, it is easy to check that the integral

$$\int_0^T f(s)\partial_s H_n(s)\,ds$$

is of order $n\big[f(a^+)-f(a^-)\big]$. These two facts imply (4.35), proving the claim.

The proof that the statement is a straightforward adaptation of the claim above, and details are omitted here. □

Proposition 4.4.5. *The set of profiles $\rho\in C\big([0,T]\times\mathbb{T}\big)$ such that $\rho\geq\varepsilon>0$ for some $\varepsilon=\varepsilon(\rho)>0$ is $\boldsymbol{I}$-dense.*

Proof. If $\rho_0\in\mathcal{D}\big([0,T],C(\mathbb{T})\big)$ is such that $\boldsymbol{I}(\rho_0)<\infty$, we known by Proposition 4.4.4 that $\rho\in C\big([0,T]\times\mathbb{T}\big)$. Let $\rho_n=\frac{1}{n}+\big(1-\frac{1}{n}\big)\rho_0$, which converges to ρ_0 as $n\to\infty$. Since $\boldsymbol{I}$ is l.s.c, then

$$\liminf_{n\to\infty}\boldsymbol{I}(\rho_n)\;\geq\;\boldsymbol{I}(\rho_0)\,.$$

Since J^2 is continuous, then

$$\lim_{n\to\infty}J^2(\rho_n)\;=\;J^2(\rho_0)\,.$$

And since $\boldsymbol{I}^1$ is convex, then

$$\limsup_{n\to\infty}\boldsymbol{I}(\rho_n)\;\leq\;\limsup_{n\to\infty}\frac{1}{n}\boldsymbol{I}(1)+\limsup_{n\to\infty}\big(1-\frac{1}{n}\big)\boldsymbol{I}(\rho_0)\;=\;\boldsymbol{I}(\rho_0)\,.$$

Therefore, $\lim_{n\to\infty}\boldsymbol{I}(\rho_n)=\boldsymbol{I}(\rho_0)$. □

Proposition 4.4.6. *The set of profiles $\rho\in C^{0,\infty}\big([0,T]\times\mathbb{T}\big)$ such that $\rho\geq\varepsilon>0$ for some $\varepsilon=\varepsilon(\rho)>0$ is $\boldsymbol{I}$-dense.*

Proof. By the Proposition 4.4.5, it is enough to prove the $\boldsymbol{I}$-density of the set above on the set of profiles $\rho \in C\big([0,T]\times\mathbb{T}\big)$ such that $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$. Let $\Psi_\delta : \mathbb{T} \to \mathbb{R}$ be an approximation of identity, that is, $\int_{\mathbb{T}} \Psi_\delta(x)dx = 1$, $\Psi_\delta \geq 0$, $\operatorname{supp}(\Psi_\delta) \subset (-\delta,\delta)$, Ψ_δ is symmetric around zero and $\Psi \in C^\infty(\mathbb{T})$. Denote by $(\Psi_\delta * \rho)(t,u)$ the spatial convolution of Ψ_δ with $\rho \in C\big([0,T], C^\infty(\mathbb{T})\big)$.

It is simple to check that $\Psi_\delta * \rho$ converges to ρ as $\delta \searrow 0$. Thus, by Proposition 4.4.3,

$$\lim_{\delta\to 0} J(\Psi_\delta * \rho) \;=\; J(\rho)\,, \tag{4.36}$$

and

$$\liminf_{\delta\to 0} \boldsymbol{I}^1(\Psi_\delta * \rho) \;\geq\; \boldsymbol{I}^1(\rho)\,.$$

On the other hand, since $\boldsymbol{I}^1$ is convex and (spatially) translation invariant, we get that

$$\boldsymbol{I}^1(\Psi_\delta * \rho) \;\leq\; \int_{\mathbb{T}} \boldsymbol{I}^1(T_u\rho)\Psi_\delta(u)\,du \;=\; \int_{\mathbb{T}} \boldsymbol{I}^1(\rho)\Psi_\delta(u)\,du \;=\; \boldsymbol{I}^1(\rho)\,,$$

where T_u denotes the rotation of u on the torus $\mathbb{T}$. Thus $\limsup_{\delta\to 0} \boldsymbol{I}^1(\Psi_\delta*\rho) \leq \boldsymbol{I}^1(\rho)$, which leads us to

$$\lim_{\delta\to 0} \boldsymbol{I}^1(\Psi_\delta * \rho) \;=\; \boldsymbol{I}^1(\rho)\,. \tag{4.37}$$

Putting together (4.36) and (4.37) concludes the proof. □

Proposition 4.4.7. *The set of profiles $\rho \in C^{\infty,\infty}\big([0,T]\times\mathbb{T}\big)$ such that $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$ is $\boldsymbol{I}$-dense.*

Proof. By Proposition 4.4.6, it is enough to assure the $\boldsymbol{I}$-density on the set of profiles $\rho \in C^{0,\infty}\big([0,T]\times\mathbb{T}\big)$ such that $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$. Let henceforth be ρ with these properties and such that $\boldsymbol{I}(\rho) < \infty$.

Let $\Psi_{1/n}$ be a time-approximation of identity, that is, $\Psi_{1/n}$ has support in $(-1/n, 1/n)$, is symmetric around zero, non negative and $C^\infty(\mathbb{R})$. We define now a suitable kind of time translation. Set, for $t \in [0,T]$,

$$\sigma_t\rho(s,u) \;=\; \begin{cases} \rho(s+t,u) & \text{for } 0 \leq s \leq T-t,\\ \rho(T,u) & \text{for } T-t \leq s \leq T,\end{cases}$$

and set, for $t \in [-T,0]$,

$$\sigma_t\rho(s,u) \;=\; \begin{cases} \rho(s+t,u) & \text{for } -t \leq s \leq T,\\ \rho(0,u) & \text{for } 0 \leq s \leq -t.\end{cases}$$

For $n \in \mathbb{N}$ such that $1/n < T/2$, let

$$\rho_n(t,u) \;=\; \int_{-T}^{T} \Psi_{1/n}(s)\sigma_s\rho(t,u)ds\,.$$

It is easy to check that ρ_n converges to ρ, thus $J(\rho_n)$ converges to $J(\rho)$ as $n \to \infty$. By the convexity of $\boldsymbol{I}^1$ and an adaptation of [11, Prop. 3.1], we get that

$$\boldsymbol{I}^1(\rho_n) \;\leq\; \boldsymbol{I}^1(\rho) + \frac{c}{n}\,,$$

where $c = c(\rho)$ is a constant. This inequality and the lower semi-continuity of $\boldsymbol{I}^1$ implies that $\lim_{n\to\infty} \boldsymbol{I}(\rho_n) = \boldsymbol{I}(\rho)$, concluding the proof. □

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Universidade Federal de Minas Gerais - UFMG
Departamento de Matemática / Programa de pós-graduação em Matemática

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Av. Pres. Antônio Carlos, 6627 - Pampulha, Belo Horizonte - MG
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