UNIVERSIDADE FEDERAL DA BAHIA - UFBA
INSTITUTO DE MATEMÁTICA
E ESTATÍSTICA - IME
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA - PGMAT
TESE DE DOUTORADO

THE SLOW BOND RANDOM WALK AND THE SNAPPING OUT BROWNIAN MOTION

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Salvador-Bahia
Julho de 2020

The Slow Bond Random Walk and the Snapping Out Brownian Motion

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Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal da Bahia como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática.

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Salvador-Bahia
Julho de 2020

Sistema de Bibliotecas da UFBA

da Silva, Diogo Soares Dórea
The Slow Bond Random Walk and the Snapping Out Brownian Motion/ Diogo Soares Dórea da Silva. Salvador – 2020.
51 f. : il

Orientador: Prof. Dr. Tertuliano Franco Santos Franco.
Tese (Doutorado - Doutorado em Matemática) – Universidade Federal da Bahia, Instituto de Matemática, Salvador, 2020.

1. Central Limit Theorem. 2. Slow Bond Random Walk. 3. Snapping Out Brownian Motion. I. Franco, Tertuliano Franco Santos. II. Universidade Federal da Bahia. Instituto de Matemática. III. Título.

CDD - 519.2
CDU - 519

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Tese submetida ao corpo docente da pós-graduacão e pesquisa do Instituto de Matemática e Estatística da Universidade Federal da Bahia como parte dos requisitos necesários para a obtenção do grau de doutor em Matemática.

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A Helen e à minha família.

Agradecimentos

Agradeço aos meus pais, que sempre foram referências de amor, dedicação e humanidade para mim. Pela educação, noção de civilidade e consciência que proporcionaram desde sempre. Por estarem juntos a mim, mesmo que não fisicamente, acompanhando a minha trajetória.

Agradeço à minha família: irmã, tias, tios, primas, primos e avós, que mesmo espalhada em várias partes do país, vem sempre mantendo contato, apoiando minha jornada.

Agradeço imensamente à minha companheira Helen. Pela diversidade de coisas que vivenciamos juntos, estando sempre ao meu lado, apoiando-me, confortando-me, ouvindo-me e fazendo-nos felizes nessa jornada. Por compartilhar tantos sorrisos e momentos de alegria desde o começo da nossa graduação. Pela paciência nos momentos em que estive ausente. Pelo amor e cuidado que transmitiu para mim, mesmo quando não os retribuí. Pelo companheirismo e por não ter-me deixado só nos momentos de dificuldade. Por ser um exemplo de caráter, com o qual procuro espelhar-me muitas das vezes.

Sou grato aos pais e irmão de Helen, por receberem-me e acolherem-me em Salvador. Agradeço pelo suporte e carinho dispensados a mim em todo esse tempo.

Agradeço aos muitos colegas que passaram pela pós-graduação e com quem pude compartilhar as angústias e alegrias nesta fase espinhosa, mas doce, da vida acadêmica, sobretudo ao grupo Família dos Bons. São eles Careca, Danilinho, Danúzia, Edvan, Formigão, Kinder, Jacqueline, Jiló, Pedroca, Rasta e Vital. Com eles, compartilho a inquietude de mentes que não se dobram às grandes doses de obscurantismo e retrocesso que nos cercam.

Sou grato a Marcelo Lino e Roque, colegas do IFBA, pela companhia, amizade e por tornarem a jornada para Valença menos cansativa e mais divertida.

Agradeço aos meus dois orientadores, exemplos de grandes homens, humanos e profissionais, com os quais tive a honra poder ter trabalhado junto.

Ao meu orientador Tertu, por toda a dedicação e afinco com o qual desenvolve seu trabalho de professor e pesquisador. Sempre solícito, até mesmo quando não havia mais tempo, buscou ajudar da melhor forma, das disciplinas iniciais à conclusão da tese. Obrigado pela paciência, até mesmo nos momentos que abusei via whatsapp durante essa fase de isolamento social. Os 8 anos como orientando e aluno de Tertu, desde a graduação, foram muito enriquecedores para mim.

Ao meu coorientador Dirk, que também ajudou-me imensamente neste trabalho. Sou muito agradecido pela disponibilidade e paciência dispensadas a mim, até mesmo quando em dúvidas um pouco estapafúrdias da minha parte. Bastante grato também sou às palavras de conforto quando precisava.

Agradeço aos professores Guilherme, Marcelo, Renato e Vitor por aceitarem compor a banca e pelas sugestões, melhorias e correções apontadas. Ao professor Vitor, por ser um professor com o qual mantive várias conversas prolíficas sobre uma infinidade de coisas, e com quem tive o prazer de ter sido aluno. A Guilherme, que me ajudou bastante em alguns momentos de apuros no fechamento deste trabalho, ouvindo-me e aconselhando-me.

Por fim, agradeço à CAPES pelo suporte financeiro.

"Do rio que tudo arrasta se diz que é violento. Mas ninguém diz violentas as margens que o comprimem."

(Bertolt Brecht)

Resumo

Consideramos o passeio aleatório simétrico em tempo contínuo e com elo lento em $\mathbb{Z}$, cujas taxas são iguais a $1/2$ para todos os elos, exceto para o qual une os vértices $\{-1, 0\}$, cuja taxa associada é dada por $\alpha n^{-\beta}/2$, onde $\alpha > 0$ e $\beta \in [0, \infty]$ são os parâmetros do modelo. Provamos aqui um teorema central do limite funcional para esse passeio: se $\beta \in [0, 1)$, então ele converge para o movimento browniano usual. Se $\beta \in (1, \infty]$, converge para o movimento browniano refletido. E no valor crítico $\beta = 1$, converge para o *snapping out Brownian Motion* (SNOB) com parâmetro $\kappa = 2\alpha$, que é um processo do tipo browniano construído recentemente em [23]. Também fornecemos estimativas de Berry-Esseen na métrica dual limitada de Lipschitz para a convergência fraca de distribuições unidimensionais, que acreditamos ser finas. Ademais, são apresentados, no último capítulo, possíveis trabalhos futuros e dificuldades encontradas para obter a propagação do equilíbrio local para o *modelo de Kipnis-Marchioro-Presutti* (KMP) *com elo lento*.

Palavras-chave: Teorema Central do Limite, Passeio Aleatório com Elo Lento, *Snapping Out Brownian Motion*.

Abstract

We consider the continuous time symmetric random walk with a slow bond on $\mathbb{Z}$, whose rates are equal to $1/2$ for all bonds, except for the bond of vertices $\{-1, 0\}$, whose associated rate is $\alpha n^{-\beta}/2$, where $\alpha > 0$ and $\beta \in [0, \infty]$ are the parameters of the model. We prove a functional central limit theorem for this random walk: if $\beta \in [0, 1)$, then it converges to the usual Brownian motion, if $\beta \in (1, \infty]$, then it converges to the reflected Brownian motion, and at the critical value $\beta = 1$, it converges to the *snapping out Brownian motion* (SNOB) of parameter $\kappa = 2\alpha$, a Brownian type-process recently constructed in [23]. We also provide Berry-Esseen estimates in the dual bounded Lipschitz metric for the weak convergence of one-dimensional distributions, which we believe to be sharp. Furthermore, in the last chapter, are presented possible future works and difficulties faced to obtain the propagation of the local equilibrium for the *slow bond Kipnis-Marchioro-Presutti* (KMP) *Model*.

Keywords: Central Limit Theorem, Slow Bond Random Walk, Snapping Out Brownian Motion.

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Chapter 1

Introduction

Arguably one of the most important results in probability theory and statistical mechanics is Donsker's theorem which establishes a link between two key objects in the field: random walk and Brownian motion.

In the literature many Donsker-type theorems can be found; however, most of the results are concerned with limits of random walks (in random environment, in non-Markovian setting, in deterministic non-homogeneous medium etc) towards the *usual Brownian motion*. A significant smaller set of results are about convergence towards Brownian motion with boundary conditions, see [1] for an example.

In this work, we prove a functional central limit theorem for the *slow bond random walk* (abbreviated *slow bond RW*), which is the continuous time nearest neighbour random walk on $\mathbb{Z}$ with jump rates given by $\alpha/(2n^\beta)$ if the jump is along the edge $\{-1, 0\}$ and $1/2$ otherwise.

The jump rates of the slow bond RW are depicted in Figure 1.1. We remark that this process was inspired by the *exclusion process with a slow bond*, see [8, 11, 12, 13, 14, 15] among others. The slow bond RW can be seen simply as the *symmetric exclusion process with a slow bond* with a single particle. For the symmetric exclusion process with a slow bond, under certain initial conditions, [11, 12, 13] established a dynamical phase transition in β. Surprisingly the proof of that transition neither implies or uses a similar transition for the slow bond RW nor does it give any indication of how to establish such a result. Yet, it would be natural to expect a dynamical phase transition for the slow bond RW as well. This is exactly the content of this work.

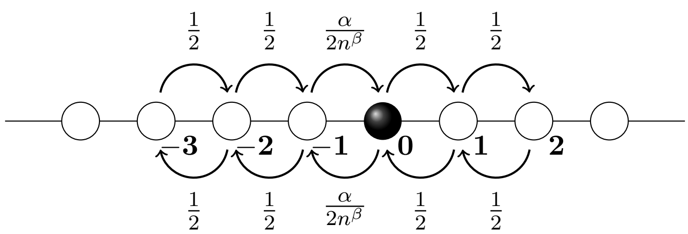

Figure 1.1: Jump rates for the *slow bond random walk*

It was shown here that the limit for the slow bond RW depends on the range of β. If $\beta \in [0,1)$, the limit is the usual Brownian motion (BM), meaning that the slow bond has no effect in the limit; if $\beta \in (1,\infty]$ it is the reflected Brownian motion, meaning that the slow bond is powerful enough to completely split the real line around the origin in the limit. Finally, and most important, in the critical case $\beta = 1$, the limit is given by the *snapping out Brownian motion*, which is a stochastic process recently constructed in [23]. This process can be understood as a Brownian motion with the following boundary behaviour: until the moment that the local time at zero reaches a value given by an (independent of the BM) exponential random variable, the process behaves as the reflected BM. At that moment, the process is then restarted, according to a fair coin, in the positive or in the negative half line (at the origin). A precise definition is given in Chapter 2 as well as a brief explanation of why the snapping out BM is related with the *partially reflected BM*, see [16] on the latter process.

A heuristic interpretation for the existence of a phase transition for $\beta = 1$ is as follows: first, it is already known that the random walk at time n is, roughly speaking, spread over a box of size $\sqrt{n}$. Therefore, on the diffusive scaling, the random walk is spread through a box of size $\sqrt{n^2} = n$. Thus, each site of that box is visited on average $1/n$ of time, which gives us a time $n^2/n = n$. In particular, the time spent at the origin is of order n and the average number of crossings over the bond $\{-1,0\}$ is of order n/n^β. This explains why a critical behavior occurs at $\beta = 1$. In this case, the number of crossings over the bond $\{-1,0\}$ is of order of a constant.

The partially reflected BM is known to be relevant in many physical situations, including nuclear magnetic resonance, heterogeneous catalysis and electric transport in electrochemistry, see [16, 17], and the same importance is expected for the snapping out BM. Some methods of simulations for both the snapping out BM and the partially reflected BM have been described, see [23, Section 6] and [16, Subsection 1.1.4] and references therein. However, no rigorous functional central limit theorem has been proved until now. Furthermore, the choice of an approximating model itself was open. Here was presented a very simple discrete model which rigorously can be shown to converge to the snapping out BM.

A relevant feature of this work is the approach itself: since the slow bond RW cannot be written as a sum of independent random variables, is not clear how to use classical approaches as convergence of characteristic functions, successive replacements (as in [3, p. 42] for instance) or via the d_k distance (see [2, Chapter 2] for instance). To overcome this difficulty, we deal directly with the convergence of expectation of bounded continuous functions to show the convergence of the one-dimensional distributions. The problem is then translated into a convergence of solutions of a semi-discrete scheme by looking at Kolmogorov's equation for the generator.

Convergence of semi-discrete schemes with boundary conditions are often technically very challenging. However, standard techniques of convergence for these problems have been avoided. Instead, via the Feynman-Kac formula, we are able to establish convergence of the semi-discrete scheme by means of probabilistic tools. The key observation is that it is possible to rewrite the problem in terms of a simple random walk and a tilted reflected random walk. The main tools developed and used involve local times, projection of Markov chains, local central limit theorems and symmetry arguments.

The convergence of the finite dimensional distributions turn out to follow more or less directly from the convergence of the one-dimensional distributions. Tightness issues have been handled through an appropriate application of the Burkholder-Davis-Gundy inequality to the Dynkin martingale.

En passant, it was obtained, in Chapter 3 an explicit formula for the semigroup of the snapping out BM and characterize it as a solution of a PDE with Robin boundary conditions, which is a small ingredient in the proof, but of interest by itself. A substantial part of this work is dedicated to show Berry-Esseen estimates for the one-dimensional distributions in the dual bounded Lipschitz metric. The convergence rates are indeed slower than in the classical case. A discussion of why this phenomena occurs is presented in Chapter 3.

We believe that the approach of this work could be successful in other situations, in particular to prove functional central limits of random walks in non-homogeneous medium. The philosophy behind this work is that *analytical problems inherited from probabilistic problems are easier solved by probabilistic methods*.

The outline of the thesis is the following: In Chapter 2 are presented definitions and stated results. Chapter 3 is dedicated to present the semigroup formula for the snapping out BM. Chapter 4 deals with necessary ingredients in the proof of convergence of one-dimensional distributions and Berry-Esseen estimates, all of them related to local times. Chapter 5 gives the proof of Berry-Esseen estimates in the dual Lipschitz bounded norm and convergence of one-dimensional distributions. Chapter 6 extends the proof to finite-dimensional distributions and proves the tightness of the processes in the Skorohod's J_1-topology of $\mathscr{D}([0,1],\mathbb{R})$, the space of càdlàg functions on $[0,1]$, as presented in [3, p. 109] for instance. Chapter 7 show some possible continuations for this work, as well as the difficulties which can be appear. The slow bond KMP is exhibit, as well as the conjecture that it has a phase transition in the propagation of local equilibrium. Moreover, are presented some techniques that can help to prove this conjecture. In Appendix A we review some known results for the sake of completeness.

The initial part of this work was accepted for publication in the Journal Annals of Applied Probability, see [9]. We are trying to continue this work, as will be shown in Chapter 7, but we do not know if we will succeed.

Chapter 2

Statements of results

This chapter is devoted to presenting the initial definitions and notions, which will be useful in the course of the work, as well as stating the main results of the thesis. Before, we will present some important notations that were adopted.

Notation: to avoid overload notation, expectations of any process considered in this thesis starting from a point x will be denoted by $\mathbb{E}_x$. Throughout the work, the symbol $\lesssim$ will mean that the quantity standing on the left hand side of it is smaller than some multiplicative constant times the quantity on the right hand side of it. The proportionality constant may change from one line to another, but it will never depend on the scaling parameter $n \in \mathbb{N}$.

2.1 Preliminary notions

The *slow bond random walk* defined here is the Feller process on $\mathbb{Z}$ denoted by $\{X_t^{\text{slow}} : t \geq 0\}$ whose generator L_n acts on local functions $f : \mathbb{Z} \to \mathbb{R}$ via

$$\mathsf{L}_n f(x) \;=\; \tau^n_{x,x+1}\Big[f(x+1) - f(x)\Big] + \tau^n_{x,x-1}\Big[f(x-1) - f(x)\Big]\,, \tag{2.1}$$

where

$$\tau^n_{x,x+1} \;=\; \tau^n_{x+1,x} \;=\; \begin{cases} \dfrac{\alpha}{2n^\beta}, & \text{if } x = -1, \\ 1/2, & \text{otherwise}. \end{cases}$$

For more details on Feller process, its semigroups (that will be treated hereafter), generators and adjacent properties, see [26, Subsections 1.1 and 1.2]. Note that for $\alpha = 1$ and $\beta = 0$, we obtain the usual symmetric continuous time random walk.

The *elastic (or plastic or partially reflected) Brownian motion* on $[0, \infty)$ is a continuous stochastic process which can be understood as an intermediate process between the absorbed Brownian motion and the reflected Brownian motion on $[0, \infty)$. This elastic Brownian motion can be described as the reflected Brownian motion killed at a stopping time with exponential distribution: first, for a given positive parameter κ, we toss a random variable $Y \sim \exp(\kappa)$ independent of the reflected Brownian motion; once the local time of the reflected Brownian motion at zero (that will be defined later) reaches Y, it is killed (at the origin). We refer the reader to the survey [16] for the connection of the elastic Brownian motion (in the d-dimensional setting) and its connections with mixed boundary value problems and Laplacian transport phenomena.

The *snapping out Brownian motion* process on $\mathbb{G} := (-\infty, 0^-] \cup [0^+, \infty)$ with parameter κ, abbreviated SNOB, is a Feller process recently constructed in [23] by gluing pieces of the elastic BM of parameter 2κ. Since the 2κ-elastic BM is killed, it will be restarted in 0^+ or 0^- with probability $1/2$ for each. An equivalent way of defining it is to consider the κ-elastic BM, but when the process is killed at 0^+ (equiv. 0^-), it is restarted on the opposite side 0^- (equiv. 0^+).

Let $\mathscr{C}_\mathrm{b}(\mathbb{G})$ be the set of bounded continuous functions $f : \mathbb{G} \to \mathbb{R}$, which are naturally identified with the set of bounded continuous functions $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with side limits at zero. Denote by $\mathscr{C}_0(\mathbb{G}) \subset \mathscr{C}_\mathrm{b}(\mathbb{G})$ the set of bounded, continuous functions $f : \mathbb{G} \to \mathbb{R}$ vanishing at infinity. Many statements in this work can easily be extended to far more general spaces of functions. Nevertheless, since Feller semigroups are defined in terms of $\mathscr{C}_0(\mathbb{G})$, and this is enough for our purposes, we will stick to this space.

It has been shown in [23] that the semigroup of the SNOB is given by:

Theorem 2.1.1 ([23], Proposition 2)**.** *The semigroup* $(P_t^{\text{snob}})_{t\geq 0} : \mathscr{C}_0(\mathbb{G}) \to \mathscr{C}_0(\mathbb{G})$ *of the SNOB with parameter* κ *is given by*

$$\begin{aligned} P_t^{\text{snob}} f(u) \;&= \mathbb{E}_u\Big[\Big(\frac{1+e^{-\kappa L(0,t)}}{2}\Big)\, f\big(\operatorname{sgn}(u)|B_t|\big)\Big] \\ &\quad + \mathbb{E}_u\Big[\Big(\frac{1-e^{-\kappa L(0,t)}}{2}\Big)\, f\big(-\operatorname{sgn}(u)|B_t|\big)\Big]\,, \qquad \forall\, u \in \mathbb{G}\,, \end{aligned} \tag{2.2}$$

where $\{B_t : t \geq 0\}$ *is a standard Brownian Motion starting from* $u \neq 0$ *and* $L(0,t)$ *is its local time at zero.*

Above, it is understood that $\operatorname{sgn}(u) = 1$ if $u \in [0^+, \infty)$ and $\operatorname{sgn}(u) = -1$ if $u \in (-\infty, 0^-]$. For the sake of clarity, let us briefly review the notion of local time for the BM. The *occupation measure* of $\{B_t : t \geq 0\}$ up to time instant t is the (random) measure μ_t defined by the equality

$$\mu_t(A) \;=\; \int_0^t \mathbb{1}_A(B_s)\, ds\,, \qquad \forall\, A \in \mathscr{B}\,,$$

where $\mathbb{1}_A$ is the indicator function of the set A, and $\mathscr{B}$ are the Borelian sets of $\mathbb{R}$. In [24, 25], Lévy showed that, for almost all trajectories of the BM, the measure μ_t has a density $L(u,t)$ with respect to the Lebesgue measure, that is

$$\mu_t(A) \;=\; \int_A L(u,t)\, du\,, \qquad \forall\, t \geq 0\,.$$

In [31], before the advent of stochastic calculus and based on a profound study of the structure of zeros of BM, Trotter proved that there exists a modification of $L(u,t)$ (the local time at $\mathbb{R}$ up to the time t) which is continuous on $\mathbb{R}\times[0,\infty)$. With a slight abuse of notation, we denote such a modification also by $L(u,t)$. It therefore holds with probability one that

$$L(u,t) \;=\; \lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(u-\varepsilon,u+\varepsilon)}(B_s)\,ds\,, \qquad \forall\,(u,t)\in\mathbb{R}\times[0,\infty).$$

An equivalent and elegant definition of Brownian local times by means of Itô-calculus is provided by Tanaka's formula

$$L(u,t) \;=\; |B_t-u|-|B_0-u|-\int_0^t \operatorname{sgn}(B_s)\,dB_s\,,$$

which holds for any $u\in\mathbb{R}$, see for instance [29, p. 239]. On the equivalence between these two notions of local time, see [29, p. 224, Corollary 1.6]. For some history on the development of local times and earlier references, see the survey [4], and for a more modern proof on the existence of the jointly continuous modification of the local time, see [29, p. 225, Theorem 1.7].

We comment that [23] only states that the SNOB is a strong Markov process. But the fact that the SNOB is a Feller process is a simple consequence of formula (2.2), continuity, and positiveness of $L(u,t)$, which put together imply that $P_t^{\text{snob}}\mathscr{C}_0(\mathbb{G})\subset\mathscr{C}_0(\mathbb{G})$ by the Dominated Convergence Theorem, as commented in [23].

2.2 Statements of the main results

The main result of this work consists of the following Donsker-type theorem, which is an interesting connects the *slow bond random walk* with the *snapping out Brownian motion*.

Theorem A. *Let $u\in\mathbb{R}\backslash\{0\}$ and consider the slow bond random walk $\{n^{-1}X^{\text{slow}}_{tn^2}:t\in[0,1]\}$ starting from the site $\lfloor un\rfloor\in\mathbb{Z}$. Then, $\{n^{-1}X^{\text{slow}}_{tn^2}:t\in[0,1]\}$ converges in distribution, with respect to the Skorohod's J_1-topology of $\mathscr{D}([0,1],\mathbb{R})$, to a process $Y=\{Y_t:t\in[0,1]\}$, where Y is:*

- *for $\beta\in[0,1)$, the Brownian motion B starting from u.*
- *for $\beta=1$, the snapping out Brownian motion B^{snob} of parameter $\kappa=2\alpha$ starting from u.*
- *for $\beta\in(1,\infty]$, the reflected Brownian motion B^{ref} starting from u.*

Above, it is understood that B^{ref} is the reflected Brownian motion with state space $\mathbb{G}$. The semigroup of $\{B_t:t\geq 0\}$ is

$$P_tf(u) \;=\; \mathbb{E}_u\big[f(B_t)\big] \;=\; \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}}f(y)\,dy\,, \quad \text{for any } u\in\mathbb{R}\,, \tag{2.3}$$

while the semigroup of the reflected Brownian motion is given by

$$P_t^{\text{ref}}f(u) \;=\; \begin{cases} \dfrac{1}{\sqrt{2\pi t}}\displaystyle\int_0^{+\infty}\Big[e^{-\frac{(u-y)^2}{2t}}+e^{-\frac{(u+y)^2}{2t}}\Big]f(y)\,dy\,, & \text{for } u\in[0^+,\infty)\,,\\[2ex] \dfrac{1}{\sqrt{2\pi t}}\displaystyle\int_0^{+\infty}\Big[e^{-\frac{(u-y)^2}{2t}}+e^{-\frac{(u+y)^2}{2t}}\Big]f(-y)\,dy\,, & \text{for } u\in(-\infty,0^-]\,. \end{cases}$$

The main novelty of this work is the proof of the Theorem A. As already mentioned in the introduction, we split the Kolmogorov forward equation into two equations associated with the odd and even part of the initial condition, respectively. It turns out that the equation associated to the even part coincides with the Kolmogorov forward equation of simple random walk, which then can be dealt with by standard results. On the other hand, the equation associated to the odd part can be analysed in terms of the Kolmogorov forward equation of a tilted reflected simple random walk. Using the Feynman-Kac formula and a projection technique (see Proposition 4.2.1) the problem can then be reduced to the analysis of local times. This will be executed in the Chapter 4.

Next, we present a connection between the SNOB with a partial differential equation with Robin boundary conditions, result that will be proven in next chapter. That proof uses the resolvent family of the SNOB, presented in [23], and the semigroup expression of the (2.4) solution, presented in [12].

Proposition A. *Let $(P_t^{\text{snob}})_{t\geq 0} : \mathscr{C}_0(\mathbb{G}) \to \mathscr{C}_0(\mathbb{G})$ be the semigroup of the SNOB with parameter κ. Then, for any $f \in \mathscr{C}_0(\mathbb{G})$, we have that $P_t^{\text{snob}} f(u)$ is the solution of the partial differential equation*

$$\begin{cases} \partial_t \rho(t,u) \;=\; \frac{1}{2}\Delta\rho(t,u)\,, & u \neq 0 \\ \partial_u \rho(t,0^+) \;=\; \partial_u \rho(t,0^-) \;=\; \dfrac{\kappa}{2}\big[\rho(t,0^+) - \rho(t,0^-)\big]\,, & t > 0 \\ \rho(0,u) \;=\; f(u)\,, & u \in \mathbb{R}. \end{cases} \tag{2.4}$$

Moreover, the semigroup $(P_t^{\text{snob}})_{t\geq 0} : \mathscr{C}_0(\mathbb{G}) \to \mathscr{C}_0(\mathbb{G})$ is given by

$$\begin{aligned} P_t^{\text{snob}} f(u) = \frac{1}{\sqrt{2\pi t}} \Bigg\{ & \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\, dy \\ + e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_0^{+\infty} & \Big[\big(\tfrac{z-y+\kappa t}{2t}\big) e^{-\frac{(z-y)^2}{2t}} + \big(\tfrac{z+y-\kappa t}{2t}\big) e^{-\frac{(z+y)^2}{2t}} \Big] f_{\text{odd}}(y)\, dy\, dz \Bigg\}, \end{aligned} \tag{2.5}$$

for $u > 0$ and

$$\begin{aligned} P_t^{\text{snob}} f(u) = \frac{1}{\sqrt{2\pi t}} \Bigg\{ & \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\, dy \\ - e^{-\kappa u} \int_{-u}^{+\infty} e^{-\kappa z} \int_0^{+\infty} & \Big[\big(\tfrac{z-y+\kappa t}{2t}\big) e^{-\frac{(z-y)^2}{2t}} + \big(\tfrac{z+y-\kappa t}{2t}\big) e^{-\frac{(z+y)^2}{2t}} \Big] f_{\text{odd}}(y)\, dy\, dz \Bigg\}, \end{aligned} \tag{2.6}$$

for $u < 0$, where f_{even} and f_{odd} are the even and odd parts of f, respectively.

In order to state the Berry-Esseen estimates, we review some further concepts of weak convergence on probability spaces. Given a metric space (S,d), the space of bounded Lipschitz functions $\text{BL}(S)$ is the set of real functions on S such that

$$\|f\|_\infty \;=\; \sup_{u\in S} |f(u)| \;<\; \infty\,, \quad \text{and} \tag{2.7}$$

$$\|f\|_{\mathrm{L}} \;=\; \sup_{\substack{u,v\in S\\ u\neq v}} \frac{|f(u)-f(v)|}{d(u,v)} \;<\; \infty\,. \tag{2.8}$$

$\mathrm{BL}(S)$ is a normed linear space with the norm $\|f\|_{\mathrm{BL}} = \|f\|_\infty + \|f\|_{\mathrm{L}}$. This norm is known as the *bounded Lipschitz norm*. Let $\mathscr{P}(S)$ be the set of probability measures on the measurable space $(S,\mathscr{S})$, where $\mathscr{S}$ are the Borelian sets of S. The *dual bounded Lipschitz metric* d_{BL} on $\mathscr{P}(S)$ is defined by

$$d_{\mathrm{BL}}(\mu,\nu) \;=\; \sup_{\substack{f\in \mathrm{BL}(S)\\ \|f\|_{\mathrm{BL}}\leq 1}} \Big|\int f d\mu - \int f d\nu\Big|\,. \tag{2.9}$$

Under the additional condition that (S,d) is separable, d_{BL} becomes a metric for the weak convergence. That is, given $\mu,\mu_n\in\mathscr{P}(S)$, we have that $\mu_n\Rightarrow\mu$ if, and only if, $d_{\mathrm{BL}}(\mu_n,\mu)\to 0$. See [2, p. 11, Corollary 2.5], for instance.

In this work, the metric space S above will be either $\mathbb{R}$ or $\mathbb{G}$. The metric space $\mathbb{G} = (-\infty,0^-]\cup[0^+,\infty)$ has two isolated connected components. In such a case, the supremum in (2.8) can be restricted to the pairs x,y belonging to the same connected component with no prejudice to the facts above. This will be assumed henceforth. Moreover, the set $\frac{1}{n}\mathbb{Z}$ can be embedded into both sets $\mathbb{R}$ and $\mathbb{G}$. When embedding $\frac{1}{n}\mathbb{Z}$ into $\mathbb{G}$, one must only have the caution of assuming that $\frac{0}{n}=0^+$ and to look at test functions $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ that are continuous from the right at zero.

Theorem B (Berry-Esseen estimates). *Fix $t>0$ and $u\neq 0$. Denote by $\mu^{\mathrm{slow}}_{tn^2}$ the probability measure on $\mathbb{R}$ induced by the slow bond random walk $X^{\mathrm{slow}}_{tn^2}/n$ starting from $\lfloor un\rfloor$. Moreover, denote by μ^{snob}_t and μ^{ref}_t the probability measures on $S=\mathbb{G}$ induced by B^{snob}_t and B^{ref}_t, respectively, and denote by μ_t the probability measure on $S=\mathbb{R}$ induced by the Brownian motion B_t. All the previous Brownian motions are assumed to start from u. We have that:*

- *If $\beta\in[0,1)$, then*

$$d_{\mathrm{BL}}(\mu^{\mathrm{slow}}_{tn^2},\mu_t) \;\lesssim\; n^{\beta-1}\,.$$

- *If $\beta=1$, then for any $\delta>0$,*

$$d_{\mathrm{BL}}(\mu^{\mathrm{slow}}_{tn^2},\mu^{\mathrm{snob}}_t) \;\lesssim\; n^{-1/2+\delta}\,.$$

- *If $\beta\in(1,\infty]$, then*

$$d_{\mathrm{BL}}(\mu^{\mathrm{slow}}_{tn^2},\mu^{\mathrm{ref}}_t) \;\lesssim\; \max\{n^{-1},n^{1-\beta}\}\,.$$

We comment that the convergences above are slower than the Berry-Esseen rate of convergence for the symmetric random walk, which is of order n^{-1} (keep in mind that we are considering the diffusive time scaling n^2). An intuition of why this is so is as follows.

If $\beta\in[0,1)$, the slow bond random walk converges to the usual Brownian motion. However, the slow bond hinders the passage through the origin, thus making the speed of convergence slower.

If $\beta = 1$, as we shall see, an invariance principle for local times of the reflected random walk plays a protagonist role in the proof of the result above. It is known that invariance principles for local times of the Brownian motion have speed of convergence[1] of order at most $n^{-\frac{1}{2}}$. This slower rate of convergence for local times is thus inherited by the rate of convergence for the slow bond random walk.

If $\beta \in (1, \infty]$, the convergence of the slow bond random walk is towards the reflected Brownian motion. In this case, the slow bond random walk may occasionally jump over the slow bond, being trapped with high probability in the "wrong" half line. This fact is responsible for a slower rate of convergence. Note that when $\beta \geq 2$, then $\max\{n^{-1}, n^{1-\beta}\} = n^{-1}$ and the slow bond does not interfere in the rate of convergence.

Remark 2.2.1. For the case $\beta = 1$, in view of [6] it is natural to expect that the sharpest estimate should be $n^{-1/2}$ times a logarithmic correction. We expect that it would be possible with our methods to obtain such a bound upon analysing carefully and improving existing results on approximations of Brownian local times by random walk local times. For that see in particular Proposition 4.1.1, which is a key ingredient.

[1]With respect to the diffusive scaling n^2. In the ballistic scaling n, used by many authors as [28], it of course corresponds to a rate of order $n^{-1/4}$.

Chapter 3

An expression for the SNOB semigroup

Here we prove Proposition A, that is, it will be shown that the SNOB semigroup is a solution for the heat equation with boundary condition of third (or Robin) type and, moreover, we provide an explicit formula for it. In spite of the obvious importance of having an explicit formula for the semigroup (concerning applications), we explain that its deduction, as we will see, is simply a suitable connection of results from [23] and [12]. Later, this result will be needed in the proof of the central limit theorem for the slow bond random walk.

3.1 The SNOB and the heat equation with Robin boundary condition

Denote by $(G_\lambda)_{\lambda>0}$ the resolvent family of the SNOB, which acts on $f \in \mathscr{C}_0(\mathbb{G})$ via $G_\lambda f(u) = \mathbb{E}_u\Big[\int_0^\infty e^{-\lambda t} f(B_t^{\text{snob}})dt\Big] = \int_0^\infty e^{-\lambda t} P_t^{\text{snob}} f(u)dt$. We recall the following result from [23].

Proposition 3.1.1 ([23], Proposition 1)**.** *For any $f \in \mathscr{C}_0(\mathbb{G})$, the resolvent family $(G_\lambda)_{\lambda>0}$ of the SNOB with parameter κ satisfies*

$$\Big(\lambda - \frac{1}{2}\Delta\Big)G_\lambda f(u) \;=\; f(u)\,, \quad u \in \mathbb{G}\,, \tag{3.1}$$

$$\partial_u G_\lambda f(0^+) = \partial_u G_\lambda f(0^-) = \frac{\kappa}{2}\big[G_\lambda f(0^+) - G_\lambda f(0^-)\big]\,. \tag{3.2}$$

The knowledge on the resolvent family permits to characterize the generator of a Feller process. See, for instance, [29, Exercise (1.15) page 290].

Proof of Proposition A. Denote by $(P_t^{\text{Robin}})_{t\geq 0} : \mathscr{C}_0(\mathbb{G}) \to \mathscr{C}_0(\mathbb{G})$ the semigroup determined by (2.4). That is, $P_t^{\text{Robin}} f(u)$ denotes the solution of the PDE (2.4) with initial condition $f \in \mathscr{C}_0(\mathbb{G})$. One can easily adapt the result [12, Proposition 2.3] to deduce that

$$P_t^{\text{Robin}} f(u) = \frac{1}{\sqrt{2\pi t}} \Bigg\{ \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\, dy$$

$$+ e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_0^{+\infty} \Big[(\tfrac{z-y+\kappa t}{2t}) e^{-\frac{(z-y)^2}{2t}} + (\tfrac{z+y-\kappa t}{2t}) e^{-\frac{(z+y)^2}{2t}} \Big] f_{\text{odd}}(y)\, dy\, dz \Bigg\},$$

for $u > 0$ and

$$P_t^{\text{Robin}} f(u) = \frac{1}{\sqrt{2\pi t}} \Bigg\{ \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\, dy$$

$$- e^{-\kappa u} \int_{-u}^{+\infty} e^{-\kappa z} \int_0^{+\infty} \Big[(\tfrac{z-y+\kappa t}{2t}) e^{-\frac{(z-y)^2}{2t}} + (\tfrac{z+y-\kappa t}{2t}) e^{-\frac{(z+y)^2}{2t}} \Big] f_{\text{odd}}(y)\, dy\, dz \Bigg\},$$

for $u < 0$, that corresponds to (2.5) and (2.6), respectively. A brief summary of this adaptation is given at the beginning of Appendix A for the sake of completeness.

Thus, in order to conclude the proof of Proposition A, it only remains to guarantee that $P_t^{\text{Robin}} = P_t^{\text{snob}}$.

We claim that the resolvent family $G_\lambda^{\text{Robin}} f(u) = \int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(u) dt$ for (2.4) also satisfies (3.1) and (3.2). This follows indeed from a direct computation: since P_t^{Robin} is a solution of (2.4), we have that

$$\begin{aligned}
\frac{1}{2}\Delta G_\lambda^{\text{Robin}} f(u) &= \frac{1}{2}\Delta \int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(u) dt \;=\; \int_0^\infty e^{-\lambda t} \frac{1}{2}\Delta P_t^{\text{Robin}} f(u) dt \\
&= \int_0^\infty e^{-\lambda t} \partial_t P_t^{\text{Robin}} f(u) dt \;=\; \lambda \int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(u) dt - f(u)\,,
\end{aligned}$$

which gives (3.1).

Furthermore, under the same conditions and using the same tools,

$$\begin{aligned}
\partial_u G_\lambda^{\text{Robin}} f(0^+) &= \partial_u \Big[\int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(0^+) dt \Big] \;=\; \int_0^\infty e^{-\lambda t} \partial_u P_t^{\text{Robin}} f(0^+) dt \\
&= \int_0^\infty e^{-\lambda t} \partial_u P_t^{\text{Robin}} f(0^-) dt \;=\; \partial_u \Big[\int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(0^-) dt \Big] \\
&= \partial_u G_\lambda^{\text{Robin}} f(0^-),
\end{aligned}$$

and

$$\begin{aligned}
\partial_u G_\lambda^{\text{Robin}} f(0^+) &= \partial_u \Big[\int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(0^+) dt \Big] \;=\; \int_0^\infty e^{-\lambda t} \partial_u P_t^{\text{Robin}} f(0^+) dt \\
&= \int_0^\infty e^{-\lambda t} \frac{\kappa}{2} \big[P_t^{\text{Robin}} f(0^+) - P_t^{\text{Robin}} f(0^-) \big] dt \\
&= \frac{\kappa}{2} \Big[\int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(0^+) dt - \int_0^\infty e^{-\lambda t} P_t^{\text{Robin}} f(0^-) dt \Big] \\
&= \frac{\kappa}{2} \big[G_\lambda^{\text{Robin}} f(0^+) - G_\lambda^{\text{Robin}} f(0^-) \big],
\end{aligned}$$

and (3.2) follows. This claim implies that the semigroups P_t^{Robin} and P_t^{snob} have the same infinitesimal generator. Hence they are equal, see for instance [29, page 291, Exercise 1.18]. This finishes the proof of the Proposition A. □

3.2 Boundedness in the dual bounded Lipschitz norm for the SNOB semigroup

Recall the definition of $\|\cdot\|_L$ in (2.8). For later use, we present the following corollary of Proposition A.

Corollary 3.2.1. *Let $f \in \mathscr{C}_0(\mathbb{G})$, and consider the SNOB with parameter κ. Then, for any $t > 0$, we have that $P_t^{\text{snob}} f \in BL(\mathbb{G})$ and*

$$\|P_t^{\text{snob}} f\|_{\text{BL}} \;\leq\; \|f\|_\infty\Big[1 + 2\kappa + 3\sqrt{\frac{2}{\pi t}}\Big]\,.$$

The result above is obtained basically by analytical arguments applied to bound the spatial derivative of SNOB semigroup showed in 2.5 and 2.6.

Proof. Proposition A allows to differentiate $P_t^{\text{snob}} f(u)$, by the expressions (2.5) and (2.6). In fact, for $u > 0$, we get, differentiating (2.5) with respect to u,

$$\begin{aligned}
\partial_u P_t^{\text{snob}} f(u) \;=\; & \frac{1}{\sqrt{2\pi t}}\Bigg\{ \int_{\mathbb{R}} -\tfrac{(u-y)}{t} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\, dy \\
& - \int_0^{+\infty} \Big[\big(\tfrac{u-y+\kappa t}{2t}\big) e^{\frac{-(u-y)^2}{2t}} + \big(\tfrac{u+y-\kappa t}{2t}\big) e^{\frac{-(u+y)^2}{2t}}\Big] f_{\text{odd}}(y)\, dy \\
& + \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_0^{+\infty} \Big[\big(\tfrac{z-y+\kappa t}{2t}\big) e^{-\frac{(z-y)^2}{2t}} + \big(\tfrac{z+y-\kappa t}{2t}\big) e^{-\frac{(z+y)^2}{2t}}\Big] f_{\text{odd}}(y)\, dy\, dz \Bigg\}.
\end{aligned}$$

Then, splitting the expression above in three parts, adopting hencefoth $\partial_u P_t^{\text{snob}} f(u)$ only for $u > 0$, and using the triangle inequality, we obtain

$$||\partial_u P_t^{\text{snob}} f(u)||_\infty \;\leq\; A(u) + B(u) + C(u),$$

where

$$A(u) \;=\; \Bigg\|\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} -\tfrac{(u-y)}{t} e^{-\frac{(u-y)^2}{2t}} f(y)\, dy\Bigg\|_\infty,$$

$$B(u) \;=\; \Bigg\|\frac{1}{\sqrt{2\pi t}} \int_0^{+\infty} \Big[\big(\tfrac{u-y+\kappa t}{2t}\big) e^{\frac{-(u-y)^2}{2t}} + \big(\tfrac{u+y-\kappa t}{2t}\big) e^{\frac{-(u+y)^2}{2t}}\Big] f(y)\, dy\Bigg\|_\infty,$$

and

$$C(u) \;=\; \Bigg\|\frac{1}{\sqrt{2\pi t}} \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_0^{+\infty} \Big[\big(\tfrac{z-y+\kappa t}{2t}\big) e^{-\frac{(z-y)^2}{2t}} + \big(\tfrac{z+y-\kappa t}{2t}\big) e^{-\frac{(z+y)^2}{2t}}\Big] f(y)\, dy\, dz\Bigg\|_\infty.$$

Above, we used that $||f_{\text{even}}||_\infty \;\leq\; ||f||_\infty$ and $||f_{\text{odd}}||_\infty \;\leq\; ||f||_\infty$. Moreover, by the Cauchy-Schwarz inequality and change of variables, we obtain

$$\begin{aligned}
A(u) \;&\leq\; \frac{1}{\sqrt{2\pi t}} \cdot ||f||_\infty \int_{\mathbb{R}} \tfrac{|u-y|}{t} e^{-\frac{(u-y)^2}{2t}}\, dy \\
&=\; \frac{1}{t} \cdot ||f||_\infty \cdot \int_{\mathbb{R}} |u| \frac{e^{-u^2/2t}}{\sqrt{2\pi t}}\, du \qquad (3.3)\\
&\leq\; \frac{1}{t} \cdot ||f||_\infty \cdot \frac{2t}{\sqrt{2\pi t}} \;=\; ||f||_\infty \cdot \sqrt{\frac{2}{\pi t}}.
\end{aligned}$$

Furthermore, using that the integral of a positive function increases when the integration domain increase, we can write $B(u) \le B_1(u) + B_2(u)$, where

$$B_1(u) = ||f||_\infty \int_{\mathbb{R}} \Big|\frac{u-y+\kappa t}{2t}\Big| \cdot \frac{e^{-\frac{(u-y)^2}{2t}}}{\sqrt{2\pi t}}\, dy,$$

and

$$B_2(u) = ||f||_\infty \int_{\mathbb{R}} \Big|\frac{u+y-\kappa t}{2t}\Big| \cdot \frac{e^{-\frac{(u+y)^2}{2t}}}{\sqrt{2\pi t}}\, dy.$$

Then, by change of variables, we get

$$\begin{aligned} B_1(u) &\equiv ||f||_\infty \int_{\mathbb{R}} \Big|\frac{y+\kappa t}{2t}\Big| \cdot \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy \\ &\le ||f||_\infty \int_{\mathbb{R}} \frac{|y|}{2t} \cdot \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy + ||f||_\infty \int_{\mathbb{R}} \frac{\kappa}{2} \cdot \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy \\ &= ||f||_\infty \Big[\frac{1}{\sqrt{2\pi t}} + \frac{\kappa}{2}\Big]. \end{aligned}$$

By a very similar computation, we obtain that $B_2(u) \le ||f||_\infty\big(\frac{1}{\sqrt{2\pi t}} + \frac{\kappa}{2}\big)$, and that implies that

$$B(u) \le ||f||_\infty \Big[\sqrt{\frac{2}{\pi t}} + \kappa\Big].$$

Finally, using the same arguments used in last part, we can write $C(u) \le C_1(u) + C_2(u)$, where

$$C_1(u) = \frac{||f||_\infty}{\sqrt{2\pi t}}\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_{\mathbb{R}} \Big|\frac{z-y+\kappa t}{2t}\Big| e^{-\frac{(z-y)^2}{2t}}\, dy\, dz,$$

and

$$C_2(u) = \frac{||f||_\infty}{\sqrt{2\pi t}}\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_{\mathbb{R}} \Big|\frac{z+y-\kappa t}{2t}\Big| e^{-\frac{(z+y)^2}{2t}}\, dy\, dz.$$

By change of variables, we obtain

$$\begin{aligned} C_1(u) &= ||f||_\infty\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_{\mathbb{R}} \Big|\frac{y+\kappa t}{2t}\Big| \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy\, dz \\ &\le ||f||_\infty\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_{\mathbb{R}} \frac{|y|}{2t} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy\, dz \\ &+ ||f||_\infty\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \int_{\mathbb{R}} \frac{\kappa}{2} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\, dy\, dz \\ &= ||f||_\infty\, \kappa e^{\kappa u} \int_u^{+\infty} e^{-\kappa z} \Big(\frac{1}{\sqrt{2\pi t}} + \frac{\kappa}{2}\Big)\, dz \\ &= ||f||_\infty \Big[\frac{1}{\sqrt{2\pi t}} + \frac{\kappa}{2}\Big]. \end{aligned}$$

Analogously, we obtain that $C_2(u) \leq ||f||_\infty\big(\frac{1}{\sqrt{2\pi t}} + \frac{\kappa}{2}\big)$. Then, we conclude that

$$C(u) \leq ||f||_\infty\Big[\sqrt{\frac{2}{\pi t}} + \kappa\Big].$$

Thus, we obtain

$$\|\partial_u P_t^{\text{snob}} f\|_\infty \leq A(u) + B(u) + C(u) \leq \|f\|_\infty\Big[2\kappa + 3\sqrt{\frac{2}{\pi t}}\Big]\,.$$

By the definition of the norm $||\cdot||_L$, we conclude that

$$\|P_t^{\text{snob}} f\|_{\mathrm{L}} \leq \|f\|_\infty\Big[2\kappa + 3\sqrt{\frac{2}{\pi t}}\Big]\,.$$

Note that $P_t^{\text{snob}} f(u)$ is a contraction semigroup with respect to the supremum norm:

$$\begin{aligned} ||P_t^{\text{snob}} f(u)||_\infty &= \big|\big|\mathbb{E}_u\big[f(B_t^{\text{snob}})\big]\big|\big|_\infty = \sup_{x\in\mathbb{G}} \big|\mathbb{E}_x[f(B_t^{\text{snob}})]\big| \\ &\leq \sup_{x\in\mathbb{G}} \mathbb{E}_x\big[|f(B_t^{\text{snob}})|\big] \leq \sup_{x\in\mathbb{G}} \mathbb{E}_x\big[||f||_\infty\big] = ||f||_\infty. \end{aligned}$$

Then,

$$\|P_t^{\text{snob}} f\|_{\mathrm{BL}} = \|P_t^{\text{snob}} f\|_\infty + \|P_t^{\text{snob}} f\|_{\mathrm{L}} \leq \|f\|_\infty\Big[1 + 2\kappa + 3\sqrt{\frac{2}{\pi t}}\Big]\,,$$

finishing the proof $u > 0$. For $u < 0$, one can use similar arguments. □

We remark that the well known Hölder continuity of Brownian local times (see [29, Corollary 1.8, page 226]) and (2.2) may lead to continuity in space of P_t^{snob}. However, it would not lead to the Lipschitz property above. This is reasonable: more smoothness is expected when taking averages, which cannot be deduced from pathwise continuity.

Chapter 4

Local times

In the proof of Theorem A, a joint L^1-Invariance Principle for the reflected Brownian motion and its local time (at zero) will be required, as well as some extra results about local times. This is the content of this chapter.

4.1 Local times estimates

Recall that the local time of a Brownian motion B at the point $u \in \mathbb{R}$ at time $t \geq 0$ is denoted here by $L(u,t)$. Denote by $\{X_t : t \geq 0\}$ the continuous-time symmetric simple random walk on $\mathbb{Z}$ starting from zero with jump rates $\lambda(x,y) = 1/2$ if $|x-y| = 1$ and zero otherwise, and let $\xi(x,t) = \int_0^t \mathbb{1}_{\{x\}}(X_s)ds$ be its local time at $x \in \mathbb{Z}$.

The following result shows that in some sense the pair $(X_t, \xi(0,t))$ is close with high probability to the pair $(B_t, L(0,t))$.

Proposition 4.1.1 ([5], Lemma 5.6, and [22], Theorem 3.3.3)**.** *There exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which one can define on it a continuous-time symmetric random walk X_t on $\mathbb{Z}$ and a real valued Brownian motion $\big\{B_t : t \geq 0\big\}$ such that there are positive constants $C_1 = C_1(t)$ and $C_2 = C_2(t)$ such that for any $\delta \in (0, \frac{1}{2})$, any $C > 0$ any $n \geq 1$ and any $t \geq n^{-2}$ we have the estimate*

$$\mathbb{P}\Big[\big|\xi(0,tn^2) - L(0,tn^2)\big| \geq 2t^{\frac{1}{4}+\delta}n^{\frac{1}{2}+2\delta} + C\log n\Big] \leq C_1\big(n^{\frac{1}{2}-\frac{\delta}{2}}e^{-C_2 n^\delta} + n^{1+\delta-C}\big). \tag{4.1}$$

Moreover, for the same coupling there are constants $0 < c, a < \infty$ such that, for any $\delta \in (0, 1/2]$ and any pair (t,n) as above,

$$\mathbb{P}\Big[\sup_{s\leq t} |X_{sn^2} - B_{sn^2}| \geq n^{\frac{1}{2}}\Big] \;\leq\; ce^{-an^\delta}\,. \tag{4.2}$$

We note that (4.1) was originally stated in [5, Lemma 5.6] for the discrete time random walk. In order to translate it into the continuous setting one can apply standard large deviations arguments for the number of jumps and holding times of the continuous time random walk. Using Proposition 4.1.1 above we deduce the following result.

Proposition 4.1.2. *There exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that one can define on it a continuous-time symmetric random walk X_t on $\mathbb{Z}$ and a Brownian motion $\{B_t : t \geq 0\}$ for which there is a constant $C(t) > 0$ such that, for any $\delta > 0$, any $n \geq 1$ and any $t \geq n^{-2}$,*

$$\mathbb{E}\Big[\Big|\frac{\xi(0,tn^2)}{n} - \frac{L(0,tn^2)}{n}\Big|\Big] \;\leq\; C(t)\, n^{-1/2+\delta}, \quad \textit{and} \tag{4.3}$$

$$\mathbb{E}\Big[\frac{1}{n}\big|X_{tn^2} - B_{tn^2}\big|\Big] \;\leq\; C(t)\, n^{-1/2+\delta}. \tag{4.4}$$

Proof. We only prove (4.3) since the proof of (4.4) follows the same lines of reasoning. We use the abbreviation

$$A_n \;=\; \frac{\xi(0,tn^2)}{n} - \frac{L(0,tn^2)}{n}\,.$$

Let $\delta \in (0, 1/2)$. We now write

$$\mathbb{E}\big[\,|A_n|\,\big] \;=\; \mathbb{E}\big[\,|A_n|\,\mathbb{1}_{\{|A_n|\leq 3t^{\frac{1}{4}+\delta}n^{-\frac{1}{2}+2\delta}\}}\,\big] + \mathbb{E}\big[\,|A_n|\,\mathbb{1}_{\{|A_n|> 3t^{\frac{1}{4}+\delta}n^{-\frac{1}{2}+2\delta}\}}\,\big]\,.$$

The first term on the right hand side of above is bounded by $3t^{\frac{1}{4}+\delta}n^{-\frac{1}{2}+2\delta}$. To bound the second term we apply the Cauchy-Schwarz inequality to see that

$$\mathbb{E}\big[\,|A_n|\,\mathbb{1}_{\{|A_n|> 3t^{\frac{1}{4}+\delta}n^{-\frac{1}{2}+2\delta}\}}\,\big] \;\leq\; \mathbb{E}\big[\,|A_n|^2\,\big]^{\frac{1}{2}}\,\mathbb{P}\big[\,|A_n| > 3t^{\frac{1}{4}+\delta}n^{-\frac{1}{2}+2\delta}\,\big]^{\frac{1}{2}}.$$

A direct calculation involving the usual local central limit theorem (see for instance [22, Theorem 2.5.6]) shows that the L^2-norm of $\xi(0, tn^2)/n$ is bounded in n. It is possible to adapt the proof of Proposition 4.2.4 to that end.

To assure that the same L^2-bound holds true for $L(0, tn^2)/n$, it is sufficient to note that the laws of $L(0, tn^2)/n$ and $L(0, t)$ are identical by Proposition A.2.1, and then to apply Itô's isometry. Recalling Proposition 4.1.1 concludes the proof. □

4.2 Projected Markov chain and reflected RW

The next step is to adapt the result above to the context of the reflected random walk and the reflected Brownian motion. For an illustration of the (continuous-time) reflected random walk $\{X^{\text{ref}}_t : t \geq 0\}$, see Figure 4.1.

We recall below the notion of *projection* for continuous-time Markov chains, also called *lumping* in the literature.

Proposition 4.2.1 ([8], Proposition 3.4)**.** *Let $\mathscr{E}$ be a countable set, and consider a bounded function $\zeta : \mathscr{E} \times \mathscr{E} \to [0, \infty)$. Let $\{\mathbf{Z}_t : t \geq 0\}$ be the continuous time Markov chain with state space $\mathscr{E}$ and jump rates $\{\zeta(x, y)\}_{x,y\in\mathscr{E}}$. Fix an equivalence relation $\sim$ on $\mathscr{E}$ with equivalence classes $\mathscr{E}^\sharp = \{[x] : x \in \mathscr{E}\}$ and assume that, for any $y \in \mathscr{E}$,*

$$\sum_{y'\sim y} \zeta(x, y') \;=\; \sum_{y'\sim y} \zeta(x', y') \tag{4.5}$$

whenever $x \sim x'$. Then, $\{[\mathbf{Z}_t] : t \geq 0\}$ is a Markov chain with state space $\mathscr{E}^\sharp$ and jump rates $\zeta([x], [y]) = \sum_{y'\sim y} \zeta(x, y')$.

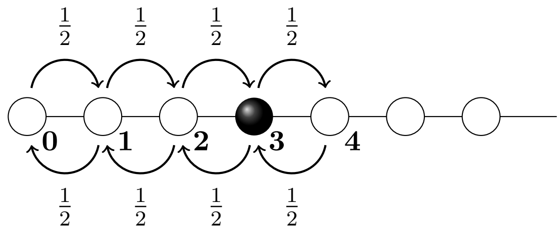

Figure 4.1: Reflected random walk on $\{0, 1, 2, \ldots\}$. All jump rates are equal to one half.

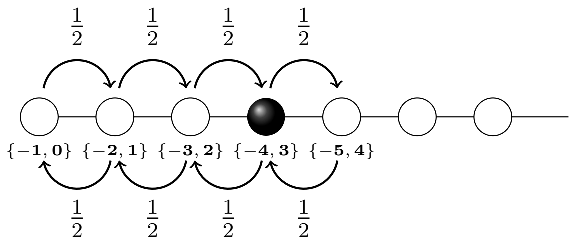

Figure 4.2: Projected Markov chain $[X_t^{\text{slow}}]$ on the state space $\Omega = \mathbb{Z}/\!\sim$. All jump rates are equal to one half.

Consider now the following equivalence relation on $\mathbb{Z}$. We will say that $x \sim y$ if, and only if,

$$x \;=\; y \quad \text{or} \quad x \;=\; -y-1\,.$$

The equivalence classes of $\mathbb{Z}/\sim$ are therefore given by $\{-1,0\},\{-2,1\}$, $\{-3,2\},\ldots$ Then, assuming that $\mathbf{Z}_t$ is the continuous-time symmetric slow bond random walk X_t^{slow} on $\mathbb{Z}$, Proposition 4.2.1 tell us that the projected Markov chain $[X_t^{\text{slow}}]$ has the rates of the reflected random walk X_t^{ref}, see Figure 4.2. Therefore, based on the construction above, we deduce that the local time at zero of the reflected random walk is almost surely equal to local time of the usual random walk on the set $\{-1,0\}$ (in this coupling).

Remark 4.2.2. Note that the usual symmetric continuous time random walk on $\mathbb{Z}$ is a particular case of X_t^{slow} taking $\beta = 0$ and $\alpha = 1$.

Remark 4.2.3. In the discrete time setting, it is true that the modulus of the symmetric random walk is the reflected random walk. However, the same does not hold in the continuous time setting, due to the fact that the waiting time at zero would be doubled when taking the modulus. This explains the choice of the equivalence relation above, which uses symmetry around the point $-1/2$.

The next result is quite intuitive, but not so immediate to prove: the times spent by the usual random walk at sites -1 and 0 are very close.

Proposition 4.2.4. *Uniformly on $x \in \mathbb{Z}$, we have the estimate*

$$\mathbb{E}_x\Big[\Big(\frac{\xi(0,tn^2)}{n} - \frac{\xi(-1,tn^2)}{n}\Big)^2\Big] \;\lesssim\; \frac{1}{n}\,. \tag{4.6}$$

In particular,

$$\mathbb{E}_x\Big[\Big|\frac{\xi(0,tn^2)}{n}-\frac{\xi(-1,tn^2)}{n}\Big|\Big]\;\lesssim\;\frac{1}{\sqrt{n}}\,. \tag{4.7}$$

Proof. First of all, observe that the function

$$f(x)\;=\;\mathbb{E}_x\Big[\Big(\frac{\xi(0,tn^2)}{n}-\frac{\xi(-1,tn^2)}{n}\Big)^2\Big]$$

is such that $f(x)\leq f(0)=f(-1)$ for any $x\in\mathbb{Z}$. The reason is simple: while the random walk does not reach 0 nor -1, both local times above remain null, which gives the inequality, while the equality is due to symmetry. Hence, let us assume without loss of generality that $x=0$. Applying the definition of the local time, a change of variables and symmetry, we obtain that

$$\begin{aligned}
&\mathbb{E}_0\Big[\Big(\frac{\xi(0,tn^2)}{n}-\frac{\xi(-1,tn^2)}{n}\Big)^2\Big]=n^2\mathbb{E}_0\bigg[\Big(\int_0^t\big(\mathbb{1}\{X_{sn^2}=-1\}-\mathbb{1}\{X_{sn^2}=0\}\big)ds\Big)^2\bigg]\\
&=\;2n^2\,\mathbb{E}_0\bigg[\int_0^t ds_1\int_0^{s_1}ds_2\Big(\mathbb{1}\{X_{s_1n^2}=X_{s_2n^2}=-1\}-\mathbb{1}\{X_{s_1n^2}=0\,,\,X_{s_2n^2}=-1\}\\
&-\mathbb{1}\{X_{s_1n^2}=-1\,,\,X_{s_2n^2}=0\}+\mathbb{1}\{X_{s_1n^2}=X_{s_2n^2}=0\}\Big)\bigg]\,.
\end{aligned}$$

Interchanging expectation and integrals and applying the Markov property, noting that $s_2\leq s_1$, the above becomes

$$\begin{aligned}
&2n^2\int_0^t ds_1\int_0^{s_1}ds_2\,\Big(\mathbb{P}_0\big[X_{s_1n^2}=X_{s_2n^2}=-1\big]-\mathbb{P}_0\big[X_{s_1n^2}=0\,,\,X_{s_2n^2}=-1\big]\\
&-\mathbb{P}_0\big[X_{s_1n^2}=-1\,,\,X_{s_2n^2}=0\big]+\mathbb{P}_0\big[X_{s_1n^2}=X_{s_2n^2}=0\big]\Big)\\
&=\;2n^2\int_0^t ds_1\int_0^{s_1}ds_2\,\Big(\mathbb{P}_0\big[X_{s_2n^2}=-1\big]\cdot\mathbb{P}_{-1}\big[X_{(s_1-s_2)n^2}=-1\big]\\
&\qquad-\mathbb{P}_0\big[X_{s_2n^2}=-1\big]\cdot\mathbb{P}_{-1}\big[X_{(s_1-s_2)n^2}=0\big]\\
&\qquad-\mathbb{P}_0\big[X_{s_2n^2}=0\big]\cdot\mathbb{P}_0\big[X_{(s_1-s_2)n^2}=-1\big]\\
&\qquad+\mathbb{P}_0\big[X_{s_2n^2}=0\big]\cdot\mathbb{P}_0\big[X_{(s_1-s_2)n^2}=0\big]\Big)\,.
\end{aligned} \tag{4.8}$$

By symmetry and translation invariance of the random walk, the integrand above can be rewritten simply as

$$\begin{aligned}
&\Big(\mathbb{P}_0\big[X_{s_2n^2}=-1\big]+\mathbb{P}_0\big[X_{s_2n^2}=0\big]\Big)\cdot\Big(\mathbb{P}_0\big[X_{(s_1-s_2)n^2}=0\big]-\mathbb{P}_0\big[X_{(s_1-s_2)n^2}=1\big]\Big)\\
&=:\;\mathbf{F}\big(s_2n^2,(s_1-s_2)n^2\big)\;=\;\mathbf{F}\,.
\end{aligned} \tag{4.9}$$

We make now some considerations on how to estimate each factor in (4.9). Let

$$p_t(x)\;\stackrel{\text{def}}{=}\;\mathbb{P}_0\big[X_t=x\big]\qquad\text{and}\qquad K_t(x)\;\stackrel{\text{def}}{=}\;\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,. \tag{4.10}$$

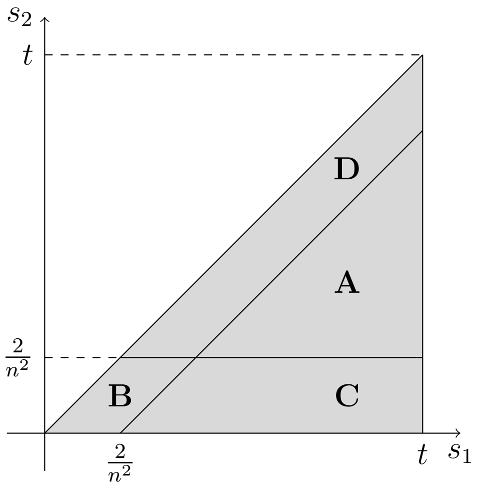

Figure 4.3: Region of integration (in gray) divided into A, B, C and D.

By the Local Central Limit Theorem (see [22, Theorem 2.5.6, p. 66]),

$$p_t(x) \;=\; K_t(x)\exp\Big\{O\Big(\frac{1}{\sqrt{t}}+\frac{|x|^3}{t^2}\Big)\Big\}$$

in the time range $t \geq 2|x|$. In particular,

$$|p_t(x)| \;\lesssim\; \frac{1}{\sqrt{t}} \quad \text{for } t \geq 2|x|\,. \tag{4.11}$$

Furthermore, adapting[1] [22, Theorem 2.3.6, p. 38] to the continuous time setting it also holds that

$$\big|p_t(x)-p_t(y)-\big(K_t(y)-K_t(x)\big)\big| \;\lesssim\; \frac{|y-x|}{t^{(d+3)/2}} \;=\; \frac{|y-x|}{t^2}\,, \tag{4.12}$$

where $d = 1$ in the current setting. We are going to use (4.12) only for $t \geq 2|y-x|$ since for all other values of t this approximation is not useful for our purposes. Noting that

$$|K_t(0)-K_t(1)| \;\leq\; \sup_{x\in[0,1]}|\partial_x K_t(x)| \;=\; \sup_{x\in[0,1]}\Big|\frac{xe^{-x^2/2t}}{t\sqrt{2\pi t}}\Big| \;\lesssim\; \frac{1}{t^{3/2}}\,,$$

we conclude that

$$\big|K_{(s_1-s_2)n^2}(0)-K_{(s_1-s_2)n^2}(1)\big| \;\lesssim\; \frac{1}{(s_1-s_2)^{3/2}n^3}\,. \tag{4.13}$$

Since the approximations (4.11) and (4.12) only hold for times $t \geq 2|x|$, we must divide the analysis of (4.8) in cases, which will be made through splitting the region of integration in disjoint sets, as depicted in Figure 4.3.

[1] As commented in ([22], p.6), the discrete time and continuous time random walks with the same increment distribution have similar behaviour. In this work, the same adaptation will be used for Theorems 2.3.5 and 2.3.11 from [22].

Region A. Here $s_2 \geq 2/n^2$ and $|s_1 - s_2| \geq 2/n^2$. Restricted to this region, both approximations (4.11) and (4.12) are valid. Recalling (4.13), we then get that

$$
\begin{aligned}
|\mathbf{F}| \;&=\; \Big(\mathbb{P}_0\big[X_{s_2n^2} = -1\big] + \mathbb{P}_0\big[X_{s_2n^2} = 0\big]\Big) \cdot \Big|\Big(\mathbb{P}_0\big[X_{(s_1-s_2)n^2} = 0\big] - \mathbb{P}_0\big[X_{(s_1-s_2)n^2} = 1\big]\Big)\Big| \\
&\lesssim\; \frac{1}{\sqrt{s_2n^2}}\Big|p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2}(1) - (K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0)) \\
&+\; (K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0))\Big| \\
&\lesssim\; \frac{1}{\sqrt{s_2n^2}}\left(\frac{1}{((s_1-s_2)n^2)^2} + \Big|K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0)\Big|\right) \\
&\lesssim\; \frac{1}{\sqrt{s_2n^2}} \cdot \left(\frac{1}{((s_1-s_2)n^2)^2} + \frac{1}{(s_1-s_2)^{3/2}n^3}\right) \\
&\lesssim\; \frac{1}{\sqrt{s_2n^2}} \cdot \frac{1}{(s_1-s_2)^{3/2}n^3} \;\lesssim\; \frac{1}{\sqrt{s_2}} \cdot \frac{1}{(s_1-s_2)^{3/2}n^4}\,,
\end{aligned}
$$

where the penultimate step is a consequence of the inequality $|s_1 - s_2| \geq 2/n^2$. Applying this bound and Fubini's Theorem we obtain that

$$
\begin{aligned}
2n^2 \iint\limits_{\mathbf{A}} ds_1\, ds_2\, |\mathbf{F}| \;&\lesssim\; n^2 \int_{2/n^2}^{t-2/n^2} ds_2 \int_{s_2+2/n^2}^{t} ds_1\, \frac{1}{\sqrt{s_2}(s_1-s_2)^{3/2}n^4} \\
&=\; \frac{2}{n^2}\int_{2/n^2}^{t-2/n^2} \frac{n\sqrt{2}}{\sqrt{s_2}} - \frac{2}{\sqrt{s_2(t-s_2)}}\, ds_2 \\
&=\; \frac{4\sqrt{2}}{n}\Big(\sqrt{t-2/n^2} - \sqrt{2}/n\Big) - \frac{8}{n^2}\Big(\arctan\Big(\sqrt{\tfrac{tn^2-2}{2}}\Big) - \arctan\Big(\sqrt{\tfrac{2}{tn^2-2}}\Big)\Big) \;\lesssim\; \frac{1}{n}\,.
\end{aligned}
$$

Region B. Here $s_2 < 2/n^2$ and $|s_1 - s_2| < 2/n^2$. Restricted to this region, neither (4.11) nor (4.12) are valid. Nevertheless, since by definition (4.9) $|\mathbf{F}| \leq 2$,

$$
2n^2 \iint\limits_{\mathbf{B}} ds_1\, ds_2\, |\mathbf{F}| \;\leq\; 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2}^{s_2+2/n^2} ds_1 \;=\; 4n^2 \cdot \frac{4}{n^4} \;\lesssim\; \frac{1}{n^2}\,.
$$

Region C. Here $s_2 < 2/n^2$ and $|s_1 - s_2| \geq 2/n^2$, where only the approximation (4.12) is valid. Using Fubini's Theorem again, the inequality $p_{s_2n^2}(-1) + p_{s_2n^2}(0) \leq 2$ and (4.12), we then have that

$$
\begin{aligned}
2n^2 \iint\limits_{\mathbf{C}} ds_1\, ds_2\, |\mathbf{F}| \;&=\; 2n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^{t} \big(p_{s_2n^2}(-1) + p_{s_2n^2}(0)\big) \\
&\quad\cdot\; \big|p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2}(1)\big|\, ds_1 \\
&\leq\; 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^{t} ds_1\, \Big|p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2}(1) \\
&-\; (K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0)) + (K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0))\Big| \\
&\leq\; 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^{t} \big|(K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0))\big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \ \left|p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2}(1) - (K_{(s_1-s_2)n^2}(1) - K_{(s_1-s_2)n^2}(0))\right| ds_1 \\
&\lesssim \ 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^t ds_1 \, \frac{1}{(s_1-s_2)^2 n^4} + \frac{1}{(s_1-s_2)^{3/2} n^3} \\
&\lesssim \ 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^t ds_1 \, \frac{2}{(s_1-s_2)^{3/2} n^3},
\end{aligned}
$$

where again the penultimate step is due to inequality $|s_1 - s_2| \geq 2/n^2$. Finally, we have

$$
\begin{aligned}
2n^2 \iint_{\mathbf{C}} ds_1 \, ds_2 \, |\mathbf{F}| \ &\lesssim \ 4n^2 \int_0^{2/n^2} ds_2 \int_{s_2+2/n^2}^t ds_1 \, \frac{2}{(s_1-s_2)^{3/2} n^3} \\
&= \ \frac{8}{n} \int_0^{2/n^2} -\frac{2}{\sqrt{t-s_2}} + n\sqrt{2} \, ds_2 \\
&= \ \frac{16}{n} \Big(2\sqrt{t - 2/n^2} - 2\sqrt{t} + \frac{\sqrt{2}}{n^2} \Big) \ \lesssim \ \frac{1}{n}.
\end{aligned}
$$

Region D. Here $s_2 \geq 2/n^2$ and $|s_1 - s_2| < 2/n^2$, where only the approximation (4.11) is valid. Using that $|p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2)}(1)| \leq 1$, we then have that

$$
\begin{aligned}
2n^2 \iint_{\mathbf{D}} ds_1 \, ds_2 \, |\mathbf{F}| \ &\leq \ 2n^2 \int_{2/n^2}^t ds_2 \int_{s_2}^{s_2+2/n^2} ds_1 \big(p_{s_2 n^2}(-1) + p_{s_2 n^2}(0) \big) \\
&\cdot \ \left| p_{(s_1-s_2)n^2}(0) - p_{(s_1-s_2)n^2)}(1) \right| \\
&\lesssim \ 2n^2 \int_{2/n^2}^t \frac{ds_2}{\sqrt{s_2 n^2}} \int_{s_2}^{s_2+2/n^2} ds_1 \\
&= \ \frac{4}{n} \int_{2/n^2}^t \frac{ds_2}{\sqrt{s_2}} \ = \ \frac{4}{n} \Big(2\sqrt{t} - \frac{2\sqrt{2}}{n} \Big) \ \lesssim \ \frac{1}{n} \, .
\end{aligned}
$$

Putting together the four estimates above gives us (4.6). Since the L^1-norm is bounded from above by the L^2-norm for probability spaces, we obtain (4.7). □

Chapter 5

CLT for a fixed time and Berry-Esseen Estimates

We begin by fixing some notation on the space of test functions.

Definition 1. *For any* $\beta \geq 0$ *we define the space* $\mathrm{BL}(\beta)$ *via*

$$\mathrm{BL}(\beta) \;=\; \begin{cases} \mathrm{BL}(\mathbb{G}), & \textit{if } \beta \in [1,\infty], \\ \mathrm{BL}(\mathbb{R}), & \textit{if } \beta \in [0,1). \end{cases} \tag{5.1}$$

Fix henceforth $f \in \mathrm{BL}(\beta)$ and denote $\frac{1}{n}\mathbb{Z} = \{\dots, -\frac{2}{n}, -\frac{1}{n}, \frac{0}{n}, \frac{1}{n}, \dots\}$. Let $g : [0,\infty) \times \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ be given by

$$g(t, \tfrac{x}{n}) \;=\; g_t\big(\tfrac{x}{n}\big) \;=\; \mathbb{E}_x\Big[f\Big(\frac{X^{\text{slow}}_{tn^2}}{n}\Big)\Big]\,. \tag{5.2}$$

Since the slow bond random walk depends on n, so does the function g, whose dependence on n has been dropped to not overload notation. Our goal is to prove the CLT directly by studying the convergence of (5.2) instead of other traditional methods, as convergence of moments, characteristics functions etc. The forward Fokker-Planck equation (for more details, see [29, p. 282]) for the generator in (2.1) then yields the semi-discrete scheme

$$\begin{cases} \partial_t g_t\big(\frac{x}{n}\big) = \frac{n^2}{2}\big[g_t\big(\frac{x+1}{n}\big) + g_t\big(\frac{x-1}{n}\big) - 2g_t\big(\frac{x}{n}\big)\big]\,, & \forall\, x \neq -1, 0 \\ \partial_t g_t\big(\frac{0}{n}\big) = \frac{n^2}{2}\big[g_t\big(\frac{1}{n}\big) - g_t\big(\frac{0}{n}\big)\big] + \frac{\alpha n^{2-\beta}}{2}\big[g_t\big(\frac{-1}{n}\big) - g_t\big(\frac{0}{n}\big)\big], & \\ \partial_t g_t\big(\frac{-1}{n}\big) = \frac{n^2}{2}\big[g_t\big(\frac{-2}{n}\big) - g_t\big(\frac{-1}{n}\big)\big] + \frac{\alpha n^{2-\beta}}{2}\big[g_t\big(\frac{0}{n}\big) - g_t\big(\frac{-1}{n}\big)\big], & \\ g\big(0, \frac{x}{n}\big) = f\big(\frac{x}{n}\big)\,, & \forall\, x \in \mathbb{Z}\,. \end{cases} \tag{5.3}$$

Note the resemblance of (5.3) above with the discrete heat equation. To continue we make some symmetry considerations. Let us consider the following notion of parity for functions $f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$, where the symmetry axis is located at $-\frac{1}{2n}$ instead of the origin. That is, we will say that $f_{\text{even}(n)} : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is an *even function* if

$$f_{\text{even}(n)}\big(\tfrac{x}{n}\big) \;=\; f_{\text{even}(n)}\big(\tfrac{-1-x}{n}\big)\,, \quad \forall x \in \mathbb{Z}\,, \tag{5.4}$$

while by an *odd function* we will mean that

$$f_{\mathrm{odd}(n)}\big(\tfrac{x}{n}\big) \;=\; -f_{\mathrm{odd}(n)}\big(\tfrac{-1-x}{n}\big)\,, \quad \forall x \in \mathbb{Z}\,. \tag{5.5}$$

The even and odd parts of a given function $f : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ are hence given by

$$f_{\mathrm{even}(n)}\big(\tfrac{x}{n}\big) \;=\; \frac{f\big(\frac{x}{n}\big) + f\big(\frac{-1-x}{n}\big)}{2} \quad \text{and} \quad f_{\mathrm{odd}(n)}\big(\tfrac{x}{n}\big) \;=\; \frac{f\big(\frac{x}{n}\big) - f\big(\frac{-1-x}{n}\big)}{2}\,, \tag{5.6}$$

and it is clear that $f\big(\frac{x}{n}\big) \;=\; f_{\mathrm{even}(n)}\big(\frac{x}{n}\big) + f_{\mathrm{odd}(n)}\big(\frac{x}{n}\big)$. Denote by $\mathbf{P}^n_t f\big(\frac{x}{n}\big)$ the solution of (5.3) with initial condition f. Due to linearity,

$$\mathbf{P}^n_t f \;=\; \mathbf{P}^n_t f_{\mathrm{even}(n)} + \mathbf{P}^n_t f_{\mathrm{odd}(n)}\,.$$

5.1 Parity invariance

Next, we argue by a simple probabilistic argument that the semi-discrete scheme (5.3) preserves parity, which is an indispensable ingredient in this work.

Proposition 5.1.1 (Parity invariance)**.** *The semigroup $\mathbf{P}^n_t$ preserves parity as defined in* (5.4) *and* (5.5)*. That is, if $h : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is even (respectively, odd), then $\mathbf{P}^n_t h$ is even (respectively, odd) for all positive times.*

Proof. By symmetry of the jump rates, the distribution of $X^{\mathrm{slow}}_{tn^2}$ starting from $x \in \mathbb{Z}$ is equal to the distribution of the stochastic process $-1 - X^{\mathrm{slow}}_{tn^2}$ with $X^{\mathrm{slow}}_{tn^2}$ starting from $-1-x$.

Suppose that $h : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is even, that is, $h\big(\frac{x}{n}\big) \;=\; h\big(\frac{-1-x}{n}\big)$. Hence

$$\begin{aligned} g\big(t, \tfrac{x}{n}\big) \;&=\; \mathbb{E}_x\Big[h\Big(\frac{X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \;=\; \mathbb{E}_{-1-x}\Big[h\Big(\frac{-1-X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \\ &=\; \mathbb{E}_{-1-x}\Big[h\Big(\frac{X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \;=\; g\big(t, \tfrac{-1-x}{n}\big)\,, \qquad \forall\, t > 0\,, \end{aligned}$$

which means that $\mathbf{P}^n_t h$ is an even function.

The argument for an odd function h is analogous: supposing that $h : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ is odd, that is, $h\big(\frac{x}{n}\big) \;=\; -h\big(\frac{-1-x}{n}\big)$, we have that

$$\begin{aligned} g\big(t, \tfrac{x}{n}\big) \;&=\; \mathbb{E}_x\Big[h\Big(\frac{X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \;=\; \mathbb{E}_{-1-x}\Big[-h\Big(\frac{-1-X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \\ &=\; -\mathbb{E}_{-1-x}\Big[h\Big(\frac{X^{\mathrm{slow}}_{tn^2}}{n}\Big)\Big] \;=\; -g\big(t, \tfrac{-1-x}{n}\big)\,, \qquad \forall\, t > 0\,, \end{aligned}$$

proving that $\mathbf{P}^n_t h$ is also an odd function. □

5.2 The even part

Let us discuss the case when (5.3) starts from $f_{\mathrm{even}(n)}$. Under our notion of parity, an even function h satisfies $h\big(\frac{-1}{n}\big) = h\big(\frac{0}{n}\big)$. This observation together with

Proposition 5.1.1 allows us to replace the factors $\alpha n^{2-\beta}/2$ appearing in (5.3) by any factor. In particular, we may replace those factors by $n^2/2$, thus concluding that $\mathbf{P}_t^n f_{\text{even}(n)}\big(\frac{x}{n}\big)$ is also a solution of

$$\begin{cases}\partial_t g_t\big(\frac{x}{n}\big) = \frac{1}{2}\Delta_n g_t\big(\frac{x}{n}\big)\,, & x \in \mathbb{Z}\,,\\ g\big(0,\frac{x}{n}\big) = f_{\text{even}(n)}\big(\frac{x}{n}\big)\,, & x \in \mathbb{Z}\,,\end{cases} \tag{5.7}$$

which is the well-known *discrete heat equation*, where $\Delta_n g(x) := n^2\big[g\big(\frac{x+1}{n}\big)+g\big(\frac{x-1}{n}\big)-2g\big(\frac{x}{n}\big)\big]$ is the discrete Laplacian. Since the discrete heat equation is also the forward Fokker-Planck equation for the symmetric random walk speeded up by n^2, we have therefore concluded that

$$\mathbf{P}_t^n f_{\text{even}(n)}\big(\tfrac{x}{n}\big) \;=\; \mathbb{E}_x\Big[f_{\text{even}(n)}\Big(\frac{X_{tn^2}}{n}\Big)\Big]\,, \tag{5.8}$$

where X_{tn^2} is the usual continuous-time symmetric random walk. Of course, now the classic central limit theorem gives us the desired convergence towards the expectation with respect to the Brownian motion B_t. There is only one detail to be handled: the notion of parity previously stated was defined on $\frac{1}{n}\mathbb{Z}$, not on $\mathbb{R}$, that is, given $f:\mathbb{R}\to\mathbb{R}$, the function $f_{\text{even}(n)}:\frac{1}{n}\mathbb{Z}\to\mathbb{R}$ as previously defined depends on the chosen value of $n\in\mathbb{N}$. Denote by $f_{\text{even}}, f_{\text{odd}}:\mathbb{R}\to\mathbb{R}$ the standard even and odd parts of f, that is,

$$f_{\text{even}}(u) \;=\; \frac{f(u)+f(-u)}{2} \quad\text{and}\quad f_{\text{odd}}(u) \;=\; \frac{f(u)-f(-u)}{2}\,,\quad \forall\, u\in\mathbb{R}\,. \tag{5.9}$$

Using the two expressions for even functions of (5.6) and (5.9), we get

$$\begin{aligned}\Big|f_{\text{even}(n)}\big(\tfrac{x}{n}\big) - f_{\text{even}}\big(\tfrac{x}{n}\big)\Big| \;&=\; \Bigg|\frac{f(\frac{x}{n})+f(\frac{-1-x}{n})}{2} - \frac{f(\frac{x}{n})+f(\frac{-x}{n})}{2}\Bigg|\\ &=\; \frac{1}{2}\Big|f\Big(\frac{-1-x}{n}\Big) - f\Big(\frac{-x}{n}\Big)\Big| \;\le\; \frac{K}{2n}\,,\qquad \forall\, x\in\mathbb{Z}\,,\end{aligned} \tag{5.10}$$

where K is the Lipschitz constant of $f\in \mathrm{BL}(\beta)$. Recall that P_t is the Brownian semigroup, as defined in (2.3). We have thus gathered the ingredients to deduce the following result:

Lemma 5.2.1. *Let $f\in\mathrm{BL}(\beta)$. Then, there exists a constant $C>0$ such that for all $t\ge 1$, all $n\in\mathbb{N}$ and all $u\in\mathbb{R}$ we have the estimate*

$$\big|\mathbf{P}_t^n f_{\text{even}(n)}\big(\tfrac{\lfloor un\rfloor}{n}\big) - P_t f_{\text{even}}(u)\big| \;\le\; \frac{C}{n}\,. \tag{5.11}$$

The result above is quite standard. However, since we did not find this exact statement in the literature, we provide a short proof of it at the end of the Appendix A.

5.3 The odd part

Let us turn our attention to the odd part. Under our notion of parity, an odd function $h : \frac{1}{n}\mathbb{Z} \to \mathbb{R}$ satisfies $h\big(\frac{-1}{n}\big) = -h\big(\frac{0}{n}\big)$. This together with the parity invariance given in Proposition 5.1.1 permits to conclude that $\mathbf{P}_t^n f_{\text{odd}(n)}\big(\frac{x}{n}\big)$, for $x \geq 0$, is a solution of

$$\begin{cases} \partial_t g_t\big(\frac{x}{n}\big) = \frac{1}{2}\Delta_n g_t\big(\frac{x}{n}\big)\,, & x \geq 1\,,\\ \partial_t g_t\big(\frac{0}{n}\big) = \frac{n^2}{2}[g_t\big(\frac{1}{n}\big) - g_t\big(\frac{0}{n}\big)] - \alpha n^{2-\beta} g_t\big(\frac{0}{n}\big)\,, & \\ g(0,\frac{x}{n}) = f_{\text{odd}(n)}\big(\frac{x}{n}\big)\,, & x \geq 0\,, \end{cases} \tag{5.12}$$

which completely determines $\mathbf{P}_t^n f_{\text{odd}(n)}$ since it is an odd function for all positive times. Define

$$L_n^{\text{ref}} f\big(\tfrac{x}{n}\big) \;=\; \begin{cases} \frac{n^2}{2}\big[f\big(\frac{x+1}{n}\big) + f\big(\frac{x-1}{n}\big) - 2f\big(\frac{x}{n}\big)\big], & x \geq 1,\\ \frac{n^2}{2}\big[f\big(\frac{1}{n}\big) - f\big(\frac{0}{n}\big)\big], & x = 0\,, \end{cases}$$

which is the generator of the reflected random walk speeded up by n^2. Writing $V_n\big(\frac{x}{n}\big) = -\alpha n^{2-\beta}\mathbb{1}_{\{0\}}(x)$, we can write (5.12) in the form

$$\begin{cases} \partial_t g_t\big(\frac{x}{n}\big) \;=\; L_n^{\text{ref}} g_t\big(\frac{x}{n}\big) + V_n\big(\frac{x}{n}\big) g\big(\frac{x}{n}\big)\,, & x \geq 0\,,\\ g\big(0,\frac{x}{n}\big) \;=\; f_{\text{odd}(n)}\big(\frac{x}{n}\big)\,, & x \geq 0\,. \end{cases}$$

The Feynman-Kac Formula, which can be found for instance in [20, p. 334, Proposition 7.1], yields that

$$\begin{aligned} \mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{x}{n}\big) \;&=\; \mathbb{E}_x\Big[f_{\text{odd}(n)}\Big(\frac{X_{tn^2}^{\text{ref}}}{n}\Big)\,\exp\Big\{\int_0^t V_n\Big(\frac{X_{sn^2}^{\text{ref}}}{n}\Big)ds\Big\}\Big]\\ &=\; \mathbb{E}_x\Big[f_{\text{odd}(n)}\Big(\frac{X_{tn^2}^{\text{ref}}}{n}\Big)\,\exp\Big\{-\alpha n^{-\beta}\xi_{tn^2}^{\text{ref}}(0)\Big\}\Big]\,, \end{aligned}$$

where

$$\xi_{tn^2}^{\text{ref}}(0) \;=\; n^2\int_0^t \mathbb{1}_{\{0\}}(x)(X_{sn^2}^{\text{ref}})\,ds \;=\; \int_0^{tn^2} \mathbb{1}_{\{0\}}(x)(X_s^{\text{ref}})\,ds$$

is the local time at zero of the reflected random walk up to time tn^2 and $X_{tn^2}^{\text{ref}}$ is the reflected randow walk speeded up by n^2. Using the coupling outlined after Proposition 4.2.1 which connects the usual symmetric random walk with the reflected random walk, and the fact that $f_{\text{odd}(n)}$ is an odd function in the sense of (5.5), we then deduce that

$$\mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{x}{n}\big) = \mathbb{E}_x\Big[f_{\text{odd}(n)}\Big(\frac{1}{n}\Big[\Big|X_{tn^2} + \frac{1}{2}\Big| - \frac{1}{2}\Big]\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}\big(\{-1,0\}\big)\Big\}\Big]. \tag{5.13}$$

Let now

$$Q_t f_{\text{odd}}(u) \;\overset{\text{def}}{=}\; \mathbb{E}_u\Big[f_{\text{odd}}\big(|B_t|\big)\exp\big\{-2\alpha L_t(0)\big\}\Big]\,, \quad \forall\, u \in \mathbb{R}\,,$$

where we recall that B_t denotes a standard Brownian motion at time t and L denotes its local time. With all these preparations at hand we can now formulate one of the main results of this chapter.

Lemma 5.3.1. *Let $f \in \mathrm{BL}(\beta)$, then for all $t > 0$ and all $u \in \mathbb{R}$ with $u > 0$ we have the estimates*

- *If $0 \leq \beta < 1$, then*

$$\big|\mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{\lfloor un \rfloor}{n}\big) - P_t f_{\text{odd}}(u)\big| \;\lesssim\; n^{\beta-1}\,.$$

- *If $\beta = 1$, then for all $\delta > 0$*

$$\big|\mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{\lfloor un \rfloor}{n}\big) - Q_t f_{\text{odd}}(u)\big| \;\lesssim\; n^{-\frac{1}{2}+\delta}\,.$$

- *If $\beta > 1$, then*

$$\big|\mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{\lfloor un \rfloor}{n}\big) - \mathbb{E}_u\big[f_{\text{odd}}(|B_t|)\big]\big| \;\lesssim\; \max\big\{n^{-1}, n^{1-\beta}\big\}\,.$$

Here the proportionality constants above depends of $\|f\|_{BL}$.

The proof of this lemma will be given in the next two sections.

5.4 Proof of Lemma 5.3.1 for $\beta \in [0,1)$

Fix $u > 0$ and $f \in \mathrm{BL}(\beta) = \mathrm{BL}(\mathbb{R})$ and recall (5.13). Furthermore, we can replace $\mathbf{P}_t^n f_{\text{odd}(n)}\big(\tfrac{\lfloor un \rfloor}{n}\big)$ by

$$\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\Big] \tag{5.14}$$

paying a price of order n^{-1}, noting that $n^{-1} \lesssim n^{\beta-1}$ for $0 \leq \beta < 1$. In fact, $\exp\big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\big\} \lesssim 1$, and

$$\Big|f_{\text{odd}(n)}\Big(\frac{|X_{tn^2}+\frac{1}{2}|}{n} - \frac{1}{2n}\Big) - f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big| \;\lesssim\; \frac{1}{n},$$

because f is Lipschitz continuous.

By the strong Markov property applied at the stopping time $T = \inf\{t \geq 0 : X_t = 0\}$, we observe now that

$$\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{X_{tn^2}}{n}\Big)\mathbb{1}_{\{T<tn^2\}}\Big] \;=\; \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{T<tn^2\}}\,\mathbb{E}_0\Big[f_{\text{odd}}\Big(\frac{X_{tn^2-T}}{n}\Big)\Big]\Big] \;=\; 0\,, \tag{5.15}$$

where the last equality follows from the facts that $\{X_{tn^2} : t \geq 0\} \overset{\text{law}}{=} \{-X_{tn^2} : t \geq 0\}$ provided $X_0 = 0$ and that f_{odd} is an odd function in the usual sense.

Thus, the term (5.14) can be rewritten as

$$\begin{aligned}&\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\Big]\\ &-\;\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big] + \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\Big]\\
&- \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\mathbb{1}_{\{T\geq tn^2\}}\Big]\\
&+ \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{X_{tn^2}}{n}\Big)\mathbb{1}_{\{T\geq tn^2\}}\Big]\\
&= \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{X_{tn^2}}{n}\Big)\Big]\\
&+ \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\mathbb{1}_{\{T< tn^2\}}\Big],
\end{aligned}
\tag{5.16}
$$

noting that, on the event $\{T \geq tn^2\}$ the exponential factor is equal to 1 and X_{tn^2} is equal to $|X_{tn^2}|$ (recall that we are assuming $u > 0$, hence not hitting 0 means not hitting -1 as well, then the local time is zero and the process is in the positive half-line). Furthermore, (5.15) guarantees that

$$
\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{X_{tn^2}}{n}\Big)\mathbb{1}_{\{T\geq tn^2\}}\Big] = \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}}\Big(\frac{X_{tn^2}}{n}\Big)\Big].
$$

The distance between $\mathbb{E}_{\lfloor un \rfloor}\big[f_{\text{odd}}\big(\frac{X_{tn^2}}{n}\big)\big]$ and $\mathbb{E}_u[f_{\text{odd}}(B_t)] = P_t f_{\text{odd}}(u)$ is bounded by some constant times $n^{-1} \lesssim n^{\beta-1}$, which can be seen exactly as in the proof of Lemma 5.2.1. Hence, in order to finish the proof of the Lemma 5.3.1 for $\beta < 1$ it is sufficient to show that the last term in (5.16) converges to zero with the desired order.

The proof that the last term in (5.16) vanishes in the limit will crucially rely on the next lemma, which may be interpreted as follows: when starting the usual random walk from $\lfloor un \rfloor$ and looking at a time window of size tn^2, either the local time (at the origin) is zero or either it is reasonably large. Since $\{T < tn^2\} = \{\xi_{tn^2}(0) > 0\}$, the situation where the local time vanishes is excluded in the second term in (5.16), which means the local time is reasonably large, which in turn yields that the exponential in the second parcel of (5.16) is reasonably small. This outlines the strategy to be followed in the sequel. Let us first state the lemma mentioned above:

Lemma 5.4.1. *Let $\gamma \in [0,1)$ and $\gamma' \in (\gamma, 1)$. Then, there is a constant $C = C(\gamma,\gamma') > 0$ such that for all $n \in \mathbb{N}$ large enough and all $j < n^{\gamma'-\gamma}$,*

$$
\mathbb{P}_{\lfloor un \rfloor}\Big[jn^\gamma < \xi_{tn^2}(0) \;\leq\; (j+1)n^\gamma\Big] \;\leq\; Cn^{\gamma-1}\,. \tag{5.17}
$$

We defer the proof of the lemma to the end of this section and we show first how it implies that the last term in (5.16) converges to zero with the desired order. Fix $\delta \in (0, 1-\beta)$. Using that $\xi_{tn^2}(\{-1,0\}) \geq \xi_{tn^2}(0)$ and $\{T < tn^2\} = \{\xi_{tn^2}(0) > 0\}$, we can then to claim that the rightmost term in (5.16) is estimated from above by $\mathrm{I} + \mathrm{II}$, where

$$
\begin{aligned}
\mathrm{I} &= \mathbb{E}_{\lfloor un \rfloor}\Big[\Big|f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big|\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(0)\Big\}\mathbb{1}_{\{\xi_{tn^2}(0)>n^{\beta+\delta}\}}\Big] \quad \text{and}\\
\mathrm{II} &= \mathbb{E}_{\lfloor un \rfloor}\Big[\Big|f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big|\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(0)\Big\}\mathbb{1}_{\{0<\xi_{tn^2}(0)\leq n^{\beta+\delta}\}}\Big].
\end{aligned}
$$

It is then straightforward to see that the term I has the desired behaviour. In fact, using that $\delta \in (0, 1-\beta)$, f_{odd} is bounded and $e^{-x} \le x+1$ for $x \ge 0$, we obtain

$$\begin{aligned} \mathrm{I} &\le \mathbb{E}_{\lfloor un \rfloor}\Big[\Big|f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big| \exp\{-\alpha n^{\beta-1}\}\Big] \\ &\le ||f_{\text{odd}}||_\infty \mathbb{E}_{\lfloor un \rfloor}\big(\exp\{-\alpha n^{\beta-1}\}\big) \lesssim n^{\beta-1}. \end{aligned}$$

To see that the same holds for II, we can also estimate it using that f_{odd} is bounded:

$$\begin{aligned} \mathrm{II} &\le \sum_{j=0}^{n^\delta} \mathbb{E}_{\lfloor un \rfloor}\Big[\Big|f_{\text{odd}}\Big(\frac{|X_{tn^2}|}{n}\Big)\Big| \exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(0)\Big\} \mathbb{1}_{\{jn^\beta < \xi_{tn^2}(0) \le (j+1)n^\beta\}}\Big] \\ &\lesssim \sum_{j=0}^{n^\delta} \exp\Big\{-\alpha j\Big\} \mathbb{P}_{\lfloor un \rfloor}\Big[jn^\beta < \xi_{tn^2}(0) \le (j+1)n^\beta\Big]. \end{aligned}$$

Applying Lemma 5.4.1 with $\gamma = \beta$ is enough to deduce the claim.

Proof of Lemma 5.4.1. We first derive the above statement for the local time of a *discrete time* symmetric random walk, which we denote by $\{S_n : n \in \mathbb{N}\}$. Moreover, denote its local time until time n of the point $a \in \mathbb{Z}$ by $\zeta_n(a)$. By [30, Equation (27)], for any $k \in \mathbb{N}$, and any $a \ge 0$ we have the formula

$$\mathbb{P}_0\big[\zeta_n(a) \ge k\big] = \mathbb{P}_0\big[S_{n-k+1} \ge a+k-1\big] + \mathbb{P}_0\big[S_{n-k+1} > a+k-1\big]. \tag{5.18}$$

Note that by symmetry the same formula applies to the local time at zero provided that $S_0 = a$. Denoting as above by T the first hitting time of zero and applying the strong Markov property at time T, we therefore see that for any $k \ge 2$,

$$\mathbb{P}_{\lfloor un \rfloor}\big[1 \le \zeta_{n^2}(0) < k\big] = \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{T<n^2\}} \mathbb{P}_0[1 \le \zeta_{n^2-T}(0) < k]\Big]. \tag{5.19}$$

Note that under $\mathbb{P}_0$ the local time at zero is always strictly positive, that is, we have $\mathbb{P}_0[\zeta_{n^2-T}(0) \ge 1] = 1$. Using (5.18) with $a = 0$ we deduce that

$$\mathbb{P}_0\big[1 \le \zeta_{n^2-T}(0) < k\big] = \mathbb{P}_0\big[S_{n^2-T-k+1} < k-1\big] - \mathbb{P}_0\big[S_{n^2-T-k+1} > k-1\big].$$

Adding and subtracting the cumulative distribution function Φ of the standard normal distribution we can write

$$\begin{aligned} &\mathbb{P}_0\big[1 \le \zeta_{n^2-T}(0) < k\big] \\ &= \mathbb{P}_0\big[S_{n^2-T-k+1} < k-1\big] - \mathbb{P}_0\big[S_{n^2-T-k+1} > k-1\big] \\ &= \mathbb{P}_0[S_{n^2-T-k+1} < k-1] + \mathbb{P}_0[S_{n^2-T-k+1} \le k-1] - 1 \\ &= \mathrm{I}(k) + \mathrm{II}(k), \end{aligned} \tag{5.20}$$

where

$$\mathrm{I}(k) = 2\Phi\Big(\frac{k-1}{\sqrt{n^2-T-k+1}}\Big) - 1 \quad \text{and}$$

$$\mathrm{II}(k) = \mathbb{P}_0[S_{n^2-T-k+1} \le k-1] + \mathbb{P}_0[S_{n^2-T-k+1} < k-1] - 2\Phi\Big(\frac{k-1}{\sqrt{n^2-T-k+1}}\Big).$$

Roughly speaking, we may say that $\mathrm{I}(k)$ and $\mathrm{II}(k)$ are close to one whenever T is close to n^2. Which would be bad, since we are aiming to show that $\mathbb{P}_0\big[1 \leq \zeta_{n^2-T}(0) < k\big]$ is small.

Therefore, to get a good upper bound on $\mathbb{P}_0\big[1 \leq \zeta_{n^2-T}(0) < k\big]$, we need to first show that the probability that T is close to n^2 is small, afterwards it remains to bound I and II for those values of T that are reasonably far away from n^2.

Let $\gamma' \in (\gamma, 1)$ as in the statement of the lemma. The Hitting Time Theorem (see [32], Theorem 1) states that

$$\mathbb{P}_{\lfloor un \rfloor}\big[T = \ell\big] \;=\; \frac{\lfloor un \rfloor}{\ell}\, \mathbb{P}_{\lfloor un \rfloor}\big[S_\ell = 0\big]\,. \tag{5.21}$$

Applying[1] the Local Central Limit Theorem [22, Theorem 2.3.5] we see that

$$\mathbb{P}_{\lfloor un \rfloor}\big[\, T \in (n^2 - n^{\gamma'}, n^2) \,\big] \;\lesssim\; un \sum_{\ell = n^2 - n^{\gamma'}}^{n^2} \frac{1}{\ell^{\frac{3}{2}}} + O\big(n^{\gamma'-2}\big)\,. \tag{5.22}$$

Here, one would actually get an extra factor $e^{-\frac{u^2 n^2}{2\ell}}$ in the sum above. Nevertheless, in the considered range of ℓ's, this factor behaves like a constant, hence it is omitted. Since $\ell \mapsto \frac{1}{\ell^{3/2}}$ is a decreasing function, we have the following inequality

$$\sum_{\ell = n^2 - n^{\gamma'}}^{n^2} \frac{1}{\ell^{\frac{3}{2}}} \;\leq\; \int_{n^2 - n^{\gamma'} - 1}^{n^2} \frac{dx}{x^{\frac{3}{2}}} \;=\; \frac{2}{\sqrt{n^2 - n^{\gamma'} - 1}} - \frac{2}{n} \;=\; O\big(n^{\gamma'-3}\big)$$

from which we can infer that the probability on the left hand side of (5.22) is of order $n^{\gamma'-2}$, which by our choice of γ' is smaller than $n^{\gamma-1}$. This provides the first ingredient of the proof, i.e., (5.19) can be estimated from above by

$$\begin{aligned}
&\mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{T < n^2 - n^{\gamma'}\}}\, \mathbb{P}_0[1 \leq \zeta_{n^2-T}(0) < k]\Big] \\
&+ \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{T \in (n^2 - n^{\gamma'}, n^2)\}}\, \mathbb{P}_0[1 \leq \zeta_{n^2-T}(0) < k]\Big] \\
&\lesssim\; \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{T < n^2 - n^{\gamma'}\}}\, \mathbb{P}_0[1 \leq \zeta_{n^2-T}(0) < k]\Big] + n^{\gamma-1}\,.
\end{aligned} \tag{5.23}$$

We turn to the analysis of $\mathrm{I}(k)$ and $\mathrm{II}(k)$. To continue, note that

$$\begin{aligned}
&\mathbb{P}_{\lfloor un \rfloor}\big[\, \zeta_{n^2}(0) \in (jn^\gamma, (j+1)n^\gamma] \,\big] \\
&= \mathbb{P}_{\lfloor un \rfloor}\big[\zeta_{n^2}(0) \in [1, (j+1)n^\gamma]\big] - \mathbb{P}_{\lfloor un \rfloor}\big[\zeta_{n^2}(0) \in [1, jn^\gamma]\big]\,,
\end{aligned}$$

which can be written in terms of differences of (5.23). Thus, in order to get the desired bounds we need to estimate $\mathrm{I}((j+1)n^\gamma) - \mathrm{I}(jn^\gamma)$ and $\mathrm{II}((j+1)n^\gamma) - \mathrm{II}(jn^\gamma)$. Using that $x \mapsto e^{-\frac{x^2}{2}}$ is decreasing in $|x|$, we see that

$$\big|\mathrm{I}((j+1)n^\gamma) - \mathrm{I}(jn^\gamma)\big| \;=\; \frac{2}{\sqrt{2\pi}} \int_{\frac{jn^\gamma - 1}{\sqrt{n^2 - T - jn^\gamma + 1}}}^{\frac{(j+1)n^\gamma - 1}{\sqrt{n^2 - T - (j+1)n^\gamma + 1}}} e^{-\frac{x^2}{2}}\, dx$$

[1] This result is stated for aperiodic processes. Nevertheless, it can be also used for odd and even times separately.

$$\lesssim \Big(\frac{(j+1)n^\gamma}{\sqrt{n^2-T-(j+1)n^\gamma}} - \frac{jn^\gamma}{\sqrt{n^2-T-jn^\gamma}}\Big)\exp\Big\{-\frac{j^2n^{2\gamma}}{2(n^2-T-jn^\gamma)}\Big\} \qquad (5.24)$$
$$\stackrel{\text{def}}{=} \mathbf{A}(j,j+1)\,.$$

Invoking (5.21), noting that $T \geq \lfloor un \rfloor$ if the random walk S starts at $\lfloor un \rfloor$, and once again recalling the Local Central Limit Theorem, we can estimate

$$\mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{0\leq T\leq n^2-n^{\gamma'}\}}\big(\mathrm{I}((j+1)n^\gamma)-\mathrm{I}(jn^\gamma)\big)\Big] \qquad (5.25)$$
$$= \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{\lfloor un \rfloor\leq T\leq n^2-n^{\gamma'}\}}\big(\mathrm{I}((j+1)n^\gamma)-\mathrm{I}(jn^\gamma)\big)\Big]$$
$$\lesssim un\sum_{k=un}^{n^2-n^{\gamma'}}\frac{1}{k^{\frac{3}{2}}}\exp\Big\{-\frac{u^2n^2}{2k}\Big\}\mathbf{A}(j,j+1)\,. \qquad (5.26)$$

To estimate the rightmost term above, we first note that for all k as above

$$\exp\Big\{-\frac{j^2n^{2\gamma}}{2(n^2-k-jn^\gamma)}\Big\} \leq \exp\Big\{-\frac{j^2n^{2\gamma}}{2(n^2-un-jn^\gamma)}\Big\}\,.$$

Writing $k=\frac{k}{n^2}n^2$, factoring out a factor n^2 of the two square root terms in (5.24), and making a Riemann sum approximation, it is a long but elementary procedure to see that (5.26) is bounded from above by some constant times

$$un^{\gamma-1}\exp\Big\{-\frac{j^2n^{2\gamma}}{2(n^2-un-jn^\gamma)}\Big\}$$
$$\times\int_{\frac{u}{n}}^{1-n^{\gamma'-2}}\frac{1}{x^{\frac{3}{2}}}\exp\Big\{-\frac{u^2}{2x}\Big\}\bigg(\frac{(j+1)}{\sqrt{1-x-(j+1)n^{\gamma-2}}}-\frac{j}{\sqrt{1-x-jn^{\gamma-2}}}\bigg)dx\,. \qquad (5.27)$$

Note that $j<n^{\gamma'-\gamma}$, thus $1-x-(j+1)n^{\gamma-2}$ is always positive in the range of x considered. Keeping this in mind one can check that u times the integral in (5.27) is uniformly bounded in n and u, therefore (5.26) is bounded by a constant times

$$n^{\gamma-1}\exp\Big\{-\frac{j^2n^{2\gamma}}{2(n^2-un-jn^\gamma)}\Big\} \lesssim n^{\gamma-1}\exp\Big\{-Cj^2n^{2(\gamma-1)}\Big\} \leq n^{\gamma-1}$$

for some constant $C>0$, which finally gives us the bound on (5.25). We now turn to the bound of II, which is easier than the previous bound for I, since there is no necessity to take differences. *Grosso modo*, we may say that

$$\mathrm{II}(k) \lesssim \frac{1}{\sqrt{n^2-T-k+1}}$$

by the usual Berry-Esseen estimate for the random walk, see [7, p. 137, Theorem 3.4.9] for instance (of course, some knowledge on T is needed to make it precise). Therefore,

$$\mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{0\leq T\leq n^2-n^{\gamma'}\}}\mathrm{II}(jn^\gamma)\Big] = \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{1}_{\{\lfloor un \rfloor\leq T\leq n^2-n^{\gamma'}\}}\mathrm{II}(jn^\gamma)\Big]$$

$$\lesssim \sum_{\ell=\lfloor un\rfloor}^{n^2-n^{\gamma'}} \mathbb{P}_{\lfloor un\rfloor}\left[T=\ell\right]\frac{1}{\sqrt{n^2-\ell-jn^\gamma+1}}\,.$$

Applying the Hitting Time Theorem, the last expression above is equal to

$$\lfloor un\rfloor \sum_{\ell=\lfloor un\rfloor}^{n^2-n^{\gamma'}} \frac{1}{\ell}\mathbb{P}_{\lfloor un\rfloor}\left[S_\ell=0\right]\frac{1}{\sqrt{n^2-\ell-jn^\gamma+1}}\,.$$

By the Local Central Limit Theorem [22, Theorem 2.3.5], the above is bounded by a constant times

$$\begin{aligned}
&\lfloor un\rfloor \sum_{\ell=\lfloor un\rfloor}^{n^2-n^{\gamma'}} \frac{1}{\ell^{3/2}}\exp\Big\{-\frac{u^2n^2}{2\ell}\Big\}\frac{1}{\sqrt{n^2-\ell-jn^\gamma+1}}\\
&\lesssim \frac{u}{n^2}\sum_{\ell=\lfloor un\rfloor}^{n^2-n^{\gamma'}} \frac{1}{(\ell/n^2)^{3/2}}\exp\Big\{-\frac{u^2}{2(\ell/n^2)}\Big\}\frac{1}{\sqrt{n^2(1-\frac{\ell-jn^\gamma+1}{n^2})}}\\
&=\frac{1}{n}\times\frac{u}{n^2}\sum_{\ell=\lfloor un\rfloor}^{n^2-n^{\gamma'}} \frac{1}{(\ell/n^2)^{3/2}}\exp\Big\{-\frac{u^2}{2(\ell/n^2)}\Big\}\frac{1}{\sqrt{(1-\frac{\ell-jn^\gamma+1}{n^2})}}\,.
\end{aligned}$$

Note now that the second factor above is a Riemann sum approximation similar to (5.27). Thus, uniformly in $j<n^{\gamma'-\gamma}$,

$$\mathbb{E}_{\lfloor un\rfloor}\Big[\mathbb{1}_{\{0\le T\le n^2-n^{\gamma'}\}}\mathrm{II}(jn^\gamma)\Big]\;\lesssim\; n^{-1}\,,$$

immediately implying that

$$E_{\lfloor un\rfloor}\Big[\mathbb{1}_{\{0\le T\le n^2-n^{\gamma'}\}}\big(\mathrm{II}(j+1)n^\gamma)-\mathrm{II}(jn^\gamma)\big)\Big]\;\lesssim\; n^{-1}\,,$$

from which the result follows for the discrete time random walk. A standard Poissonisation argument now begets the result for the continuous time case. □

5.5 Proof of Lemma 5.3.1 for $\beta\in[1,\infty]$

Proof. **Case $\beta=1$.** Using that $x\mapsto e^{-x}$ defined on $[0,\infty)$ is bounded by one and is Lipschitz continuous with Lipschitz constant one, we can estimate $|\mathbf{P}_t^n f_{\text{odd}(n)}(\frac{x}{n})-Q_tf_{\text{odd}}(\frac{x}{n})|\lesssim \mathrm{I}+\mathrm{II}$, where

$$\begin{aligned}
\mathrm{I}&=\mathbb{E}_{x,x/n}\Big[\Big|f_{\text{odd}(n)}\Big(\frac{1}{n}\Big[\Big|X_{tn^2}+\frac{1}{2}\Big|-\frac{1}{2}\Big]\Big)-f_{\text{odd}}\Big(\frac{1}{n}\big|B_{tn^2}\big|\Big)\Big|\Big]\quad\text{and}\\
\mathrm{II}&=\mathbb{E}_{x,x/n}\Big[\frac{1}{n}\big|\xi_{tn^2}(\{-1,0\})-2L_{tn^2}(0)\big|\Big]\,.
\end{aligned}$$

Here, $\mathbb{E}_{x,x/n}$ denotes the expectation induced by the coupling introduced in the proof of Proposition 4.1.2 of X and B, where both processes starts from the origin.

We first estimate I. To that end denote the Lipschitz constant of f by L (and seeing that $||f_{\text{odd}}||_\infty \leq ||f||_\infty + ||f_{\text{even}}||_\infty \leq L$, by triangle inequality), and note that for any number $a \in \mathbb{Z}$ we have the estimate $\big||a+1/2| - 1/2 - |a|\big| \leq 1$. Thus,

$$\begin{aligned}
\mathrm{I} \;&\leq\; \frac{L}{n} \times \mathbb{E}_{x,x/n}\Big[\Big|\Big(\Big|X_{tn^2} + \frac{1}{2}\Big|\Big) - \frac{1}{2} - |B_{tn^2}|\Big|\Big] \\
&\leq\; \frac{L}{n} \times \mathbb{E}_{x,x/n}\Big[\Big|\Big(\Big|X_{tn^2} + \frac{1}{2}\Big|\Big) - \frac{1}{2} - |X_{tn^2}| + |X_{tn^2}| - |B_{tn^2}|\Big|\Big] \\
&\leq\; \frac{L}{n} + \frac{L}{n} \times \mathbb{E}_{x,x/n}\Big[\Big||X_{tn^2}| - |B_{tn^2}|\Big|\Big]\,.
\end{aligned}$$

It now only remains to apply Proposition 4.1.2 to deduce directly the desired estimate for I, that is, $n^{-1/2+\delta}$. To estimate II, using triangle inequality once again, we write

$$\begin{aligned}
\mathrm{II} \;&\leq\; \mathbb{E}_{x,x/n}\left[\frac{|\xi_{tn^2}(\{-1,0\}) - 2\xi_{tn^2}(0)|}{n}\right] + 2\,\mathbb{E}_{x,x/n}\left[\frac{|\xi_{tn^2}(0) - L_{tn^2}(0)|}{n}\right] \\
&=\; \mathbb{E}_{x,x/n}\left[\frac{|\xi_{tn^2}(-1) - \xi_{tn^2}(0)|}{n}\right] + 2\,\mathbb{E}_{x,x/n}\left[\frac{|\xi_{tn^2}(0) - L_{tn^2}(0)|}{n}\right] \lesssim n^{-1/2+\delta}.
\end{aligned}$$

The result therefore follows from an application of (4.7) and (4.3), noting that $n^{-1/2} \lesssim n^{-1/2+\delta}$.

Case $\beta \in (1, \infty]$**.** We adopt the abbreviation

$$\frac{1}{n}\Big[\Big|X_{tn^2} + \frac{1}{2}\Big| - \frac{1}{2}\Big] \;=\; |X_{tn^2}|_{(n)}\,.$$

Using as above that $x \mapsto e^{-x}$ is Lipschitz continuous and bounded by 1 on $[0,\infty)$, as well as the boundedness of f (and consequently of its odd part), we see that

$$\begin{aligned}
&\Big|\mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}(n)}\Big(|X_{tn^2}|_{(n)}\Big)\exp\Big\{-\frac{\alpha}{n^\beta}\xi_{tn^2}(\{-1,0\})\Big\}\Big] - \mathbb{E}_{\lfloor un \rfloor}\Big[f_{\text{odd}(n)}\Big(|X_{tn^2}|_{(n)}\Big)\Big]\Big| \\
&\leq\; C \times \mathbb{E}_{\lfloor un \rfloor}\Big[\frac{\xi_{tn^2}(\{-1,0\})}{n^\beta}\Big] \;\leq\; Cn^{1-\beta}\,.
\end{aligned}$$

Here, we made use of Proposition A.3.1 to arrive at the last estimate. To conclude one may now proceed as in the proof of Lemma 5.2.1. □

5.6 Convergence of the slow bond random walk at a fixed time

We have gathered all ingredients to prove the main result of this section, which immediately implies Theorem B.

Theorem 5.6.1. *Let $u > 0$ and let $f \in \mathrm{BL}(\beta)$. Denote by P_t^{snob} the semigroup of the snapping out Brownian motion of parameter $\kappa = 2\alpha$. Then, for all $t > 0$, we have the estimates*

- *If* $\beta \in [0,1)$*, then*

$$\left|\mathbf{P}_t^n f\left(\tfrac{\lfloor un \rfloor}{n}\right) - P_t f(u)\right| \;\lesssim\; \max\left\{n^{-1}, n^{\beta-1}\right\} \;=\; n^{\beta-1}\,.$$

- *If* $\beta = 1$*, then for all* $\delta > 0$*,*

$$\left|\mathbf{P}_t^n f\left(\tfrac{\lfloor un \rfloor}{n}\right) - P_t^{\text{snob}} f(u)\right| \;\lesssim\; n^{-1/2+\delta}\,.$$

- *If* $\beta \in (1,\infty]$*, then*

$$\left|\mathbf{P}_t^n f\left(\tfrac{\lfloor un \rfloor}{n}\right) - \mathbb{E}_u[f(|B_t|)]\right| \;\lesssim\; \max\{n^{-1}, n^{1-\beta}\}\,.$$

Proof. **Case** $\beta \in [0,1)$. Writing $\mathbf{P}_t^n f\big(\frac{x}{n}\big) = \mathbf{P}_t^n f_{\text{odd}(n)}\big(\frac{x}{n}\big) + \mathbf{P}_t^n f_{\text{even}(n)}\big(\frac{x}{n}\big)$, we can apply Lemmas 5.2.1 and 5.3.1 to infer that $\mathbf{P}_t^n f\big(\frac{\lfloor un \rfloor}{n}\big)$ indeed converges to $P_t f_{\text{even}}(u) + P_t f_{\text{odd}}(u) = P_t f(u)$ at the desired rate.

Case $\beta = 1$. Writing $\mathbf{P}_t^n f\big(\frac{x}{n}\big) = \mathbf{P}_t^n f_{\text{odd}(n)}\big(\frac{x}{n}\big) + \mathbf{P}_t^n f_{\text{even}(n)}\big(\frac{x}{n}\big)$, Lemmas 5.2.1 and 5.3.1 imply that $\mathbf{P}_t^n f\big(\frac{\lfloor un \rfloor}{n}\big)$ converges to $P_t f_{\text{even}}(u) + Q_t f_{\text{odd}}(u)$ at the desired rate.

It therefore only remains to check that $P_t f_{\text{even}} + Q_t f_{\text{odd}} = P_t^{\text{snob}} f$, which can be verified via the following direct computation. Note that

$$f(u) + f(-u) \;=\; f(|u|) + f(-|u|)\,, \qquad \forall\, u \in \mathbb{R}\,, \tag{5.28}$$

and recall (2.2). Then,

$$\begin{aligned}
&P_t f_{\text{even}}(u) + Q_t f_{\text{odd}}(u)\\
&= \mathbb{E}_u\Big[\frac{f(B_t) + f(-B_t)}{2}\Big] + \mathbb{E}_u\Big[\frac{f(|B_t|) - f(-|B_t|)}{2}\exp\big\{-2\alpha L_t(0)\big\}\Big]\\
&= \mathbb{E}_u\Big[\frac{f(|B_t|) + f(-|B_t|)}{2}\Big] + \mathbb{E}_u\Big[\frac{f(|B_t|) - f(-|B_t|)}{2}\exp\big\{-2\alpha L_t(0)\big\}\Big]\\
&= \mathbb{E}_u\Big[\frac{1 + \exp\big\{-2\alpha L_t(0)\big\}}{2} f(|B_t|)\Big] + \mathbb{E}_u\Big[\frac{1 - \exp\big\{-2\alpha L_t(0)\big\}}{2} f(-|B_t|)\Big]\\
&= P_t^{\text{snob}} f(u)\,.
\end{aligned}$$

In the penultimate equality above, we used that $u > 0$. For $u < 0$, the adaptation is similar, and then Theorem 5.6.1 is valid also for u negative.

Case $\beta \in (1,\infty]$. It follows from the equation (2.3) and Lemmas 5.2.1 and 5.3.1 that there exists a constant $C > 0$ such that for all $t > 0$ and all n large enough

$$\left|\mathbf{P}_t^n f\left(\tfrac{\lfloor un \rfloor}{n}\right) - \mathbb{E}_u[f_{\text{even}}(B_t)] - \mathbb{E}_u[f_{\text{odd}}(|B_t|)]\right| \;\lesssim\; \max\left\{n^{-1}, n^{1-\beta}\right\}.$$

To conclude the proof it therefore only remains to show that

$$\mathbb{E}_u\big[f_{\text{even}}(B_t)\big] \;=\; \mathbb{E}_u\big[f_{\text{even}}(|B_t|)\big]\,,$$

which follows by the observation (5.28). □

Chapter 6

CLT for finite-dimensional distributions and Tightness in the J_1-Topology

6.1 CLT for finite-dimensional distributions

In what follows, since there is no necessity to specify the precise value of β, we will use B^{slow} to denote the respective limiting process in Theorem A, which can either be the BM, the snapping out BM or the reflected BM. The same applies for the notation X^{slow} for the slow bond RW.

Fix $k \in \mathbb{N}$ and times $0 = t_0 < t_1 < \ldots < t_k \le 1$. We will show in this section that

$$\Big(\frac{1}{n}X^{\text{slow}}_{t_1 n^2}, \ldots, \frac{1}{n}X^{\text{slow}}_{t_k n^2}\Big) \Longrightarrow \big(B^{\text{slow}}_{t_1}, \ldots, B^{\text{slow}}_{t_k}\big), \quad \text{as } n \to \infty, \tag{6.1}$$

where the arrow above denotes weak convergence. Let

$$\varphi_{\frac{1}{n}X^{\text{slow}}_{t_1 n^2}, \ldots, \frac{1}{n}X^{\text{slow}}_{t_k n^2}}(s_1, \ldots, s_k) \;:=\; \mathbb{E}\Big[e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2} + \ldots + s_k n^{-1} X^{\text{slow}}_{t_k n^2})}\Big], \qquad \forall\, (s_1, \ldots, s_k) \in \mathbb{R}^k$$

be the characteristic function for the joint distribution of the (properly rescaled by n^{-1}) slow bond RW at times $t_1 n^2, \ldots, t_k n^2$. In order to prove the convergence in (6.1), by ([19], Theorem 6.3) it is sufficient to prove that

$$\varphi_{\frac{1}{n}X^{\text{slow}}_{t_1 n^2}, \ldots, \frac{1}{n}X^{\text{slow}}_{t_k n^2}}(s_1, \ldots, s_k) \longrightarrow \varphi_{B^{\text{slow}}_{t_1}, \ldots, B^{\text{slow}}_{t_k}}(s_1, \ldots, s_k), \;\; \forall (s_1, \ldots, s_k) \in \mathbb{R}^k.$$

We will proceed by induction. First, handling only the times t_1 and t_2 and considering $\mathcal{F}_{t_1}$ the σ-algebra generated by process up to time $t_1 n^2$, the Markov property gives us that

$$\begin{aligned}
\varphi_{\frac{1}{n}X^{\text{slow}}_{t_1 n^2}, \frac{1}{n}X^{\text{slow}}_{t_2 n^2}}(s_1, s_2) &= \mathbb{E}_{\lfloor un \rfloor}\Big[e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2} + s_2 n^{-1} X^{\text{slow}}_{t_2 n^2})}\Big] \\
&= \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{E}_{\lfloor un \rfloor}\Big[e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2} + s_2 n^{-1} X^{\text{slow}}_{t_2 n^2})} \,|\, \mathcal{F}_{t_1}\Big]\Big]
\end{aligned}$$

$$= \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{E}_{X^{\text{slow}}_{t_1 n^2}}\big[e^{is_2 n^{-1} X^{\text{slow}}_{(t_2-t_1)n^2}}\big] e^{is_1 n^{-1} X^{\text{slow}}_{t_1 n^2}}\Big]\,. \tag{6.2}$$

Let us deal with the expectation inside parenthesis above. By Theorem 5.6.1, we know that $\mathbb{E}_k\big[e^{is_2 n^{-1} X^{\text{slow}}_{(t_2-t_1)n^2}}\big]$ is closed to $\mathbb{E}_{\frac{k}{n}}\big[e^{is_2 n^{-1} B^{\text{slow}}_{(t_2-t_1)n^2}}\big]$, uniformly in $k \in \mathbb{Z}$. We therefore conclude that (6.2) converges to

$$\mathbb{E}_u\Big[\mathbb{E}_{B^{\text{slow}}_{t_1}}\big[e^{is_2 B^{\text{slow}}_{t_2-t_1}}\big] e^{is_1 B^{\text{slow}}_{t_1}}\Big] \;=\; \varphi_{B^{\text{slow}}_{t_1}, B^{\text{slow}}_{t_2}}(s_1, s_2),$$

where the last equality is again due to the Markov property. Suppose now that

$$\varphi_{\frac{1}{n}X^{\text{slow}}_{t_1 n^2},\dots,\frac{1}{n}X^{\text{slow}}_{t_{k-1}n^2}}(s_1,\dots,s_{k-1}) \longrightarrow \varphi_{B^{\text{slow}}_{t_1},\dots,B^{\text{slow}}_{t_{k-1}}}(s_1,\dots,s_{k-1}),\;\; \forall (s_1,\dots,s_{k-1}) \in \mathbb{R}^{k-1}.$$

Thus, considering $\mathcal{F}_{t_{k-1}}$ the σ-algebra generated by process up to time $t_{k-1}n^2$, the Markov property again yields that

$$\begin{aligned}
&\varphi_{n^{-1}X^{\text{slow}}_{t_1 n^2},\dots,n^{-1}X^{\text{slow}}_{t_k n^2}}(s_1,\dots,s_k)\\
&= \mathbb{E}_{\lfloor un \rfloor}\big[e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2}+\dots+s_k n^{-1} X^{\text{slow}}_{t_k n^2})}\big]\\
&= \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{E}_{\lfloor un \rfloor}\big[e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2}+\dots+s_k n^{-1} X^{\text{slow}}_{t_k n^2})}\,|\,\mathcal{F}_{t_{k-1}}\big]\Big]\\
&= \mathbb{E}_{\lfloor un \rfloor}\Big[\mathbb{E}_{X^{\text{slow}}_{t_{k-1}n^2}}\big[e^{is_k n^{-1} X^{\text{slow}}_{(t_k-t_{k-1})n^2}}\big] e^{i(s_1 n^{-1} X^{\text{slow}}_{t_1 n^2}+\dots+s_{k-1} n^{-1} X^{\text{slow}}_{t_{k-1} n^2})}\Big]\,.
\end{aligned} \tag{6.3}$$

By the induction hypothesis and the same argument as in the previous case with two times, we deduce that (6.3) converges to

$$\mathbb{E}_u\Big[\mathbb{E}_{B^{\text{slow}}_{t_{k-1}}}\big[e^{is_k B^{\text{slow}}_{(t_k-t_{k-1})}}\big] e^{i(s_1 B^{\text{slow}}_{t_1}+\dots+s_{k-1} B^{\text{slow}}_{t_{k-1}})}\Big] = \varphi_{B^{\text{slow}}_{t_1},\dots,B^{\text{slow}}_{t_k}}(s_1,\dots,s_k)$$

concluding the proof.

6.2 Tightness in the J_1-Topology

In this section we show that the sequence $\{n^{-1}X^{\text{slow}}_{tn^2} : t \in [0,1]\}$ is tight in the Skorohod's J_1-topology of $\mathscr{D}([0,1],\mathbb{R})$. To do so, we make use of the following criterion that can be found in [3, Theorem 15.6].

Proposition 6.2.1. *Consider a sequence $(X^n)_{n\in\mathbb{N}}$ and a process X in $\mathscr{D}([0,1],\mathbb{R})$. Assume that the finite dimensional distributions of $(X^n)_{n\in\mathbb{N}}$ converge to those of X, and assume that X is almost surely continuous at $t = 1$. Moreover assume that there are $\beta \geq 0$, $\alpha > 1/2$ and a non-decreasing continuous function F such that for all $r \leq s \leq t$, all $n \geq 1$, and all $x \in \mathbb{Z}$,*

$$\mathbb{E}_x\Big[|X^n_s - X^n_r|^{2\beta}\,|X^n_t - X^n_s|^{2\beta}\Big] \;\leq\; \big[F(t) - F(r)\big]^{2\alpha}\,. \tag{6.4}$$

Then, the sequence $(X^n)_{n\in\mathbb{N}}$ converges to X in the Skorohod's J_1-topology of $\mathscr{D}([0,1],\mathbb{R})$.

As a consequence of the above result we only need to establish the moment condition (6.4). We claim that it is enough to show that there is a constant C such that for any pair of times $0 \le s \le t$, and any starting point x, the following inequality holds:

$$\mathbb{E}_x\Big[\Big|\frac{X^{\text{slow}}_{tn^2}}{n} - \frac{X^{\text{slow}}_{sn^2}}{n}\Big|^2\Big] \;\le\; C|t-s|\,. \tag{6.5}$$

Indeed assume that (6.5) holds and let $r \le s \le t$. Then the Markov property applied at time sn^2 yields

$$\begin{aligned}
&\frac{1}{n^4}\mathbb{E}_x\Big[|X^{\text{slow}}_{sn^2} - X^{\text{slow}}_{rn^2}|^2\,|X^{\text{slow}}_{tn^2} - X^{\text{slow}}_{sn^2}|^2\Big]\\
&=\;\frac{1}{n^4}\mathbb{E}_x\Big[|X^{\text{slow}}_{sn^2} - X^{\text{slow}}_{rn^2}|^2\,\mathbb{E}_{X^{\text{slow}}_{sn^2}}\big[|X^{\text{slow}}_{tn^2} - X^{\text{slow}}_{sn^2}|^2\big]\Big]\\
&\le\; C^2|t-s|\,|s-r| \;\le\; C^2|t-r|^2\,,
\end{aligned}$$

hence the claim follows. To establish (6.5), recall that Dynkin's formula yields that for any function f in the domain of L_n (as defined in (2.1)), there is a martingale $\mathscr{M}(f)$ such that

$$f\Big(\frac{X^{\text{slow}}_{tn^2}}{n}\Big) \;=\; f\Big(\frac{X^{\text{slow}}_{0}}{n}\Big) + \int_0^{tn^2} \mathsf{L}_n f\Big(\frac{X^{\text{slow}}_{s}}{n}\Big)\,ds + \mathscr{M}_{tn^2}(f)\,. \tag{6.6}$$

Our case is the case in which f is the identity. Note that in this case the definition of L_n implies that

$$\mathsf{L}_n f\Big(\frac{x}{n}\Big) \;=\; \frac{1}{n}\big[\tau_{x,x+1} - \tau_{x,x-1}\big] \;=\; \frac{\alpha}{n}\times\begin{cases}\frac{1}{2} - \frac{1}{2n^\beta}, & \text{if } x = 0,\\ \frac{1}{2n^\beta} - \frac{1}{2}, & \text{if } x = -1,\\ 0, & \text{otherwise.}\end{cases}$$

Denoting by ℓ the local time of X^{slow}, the above considerations then show that the right hand side of (6.6) equals

$$f\Big(\frac{X^{\text{slow}}_{0}}{n}\Big) + \frac{\alpha}{n}\Big[\frac{1}{2} - \frac{1}{2n^\beta}\Big]\big[\ell_{tn^2}(0) - \ell_{tn^2}(-1)\big] + \mathscr{M}_{tn^2}(f)\,.$$

Thus, by considering f as identity and using the expression above, to show (6.5) it is enough to bound the second moment of

$$\frac{1}{n}\big[\ell_{sn^2,tn^2}(0) - \ell_{sn^2,tn^2}(-1)\big] \quad\text{and}\quad \big[\mathscr{M}_{tn^2}(f) - \mathscr{M}_{sn^2}(f)\big]\,.$$

Here, we used the notation $\ell_{s,t}$ to denote the local time of X^{slow} between times s and t. We first analyse the local time term above. To that end, we note that Proposition 4.2.1 and the discussion in Remark 4.2.2 yields a coupling between $\{\ell_{tn^2}(0) + \ell_{tn^2}(-1) : t \ge 0\}$ and $\{\xi_{tn^2}(0) + \xi_{tn^2}(-1) : t \ge 0\}$ under which these processes are equal. We recall that ξ denotes the local time process of the usual continuous-time symmetric random walk. Since

$$|\ell_{sn^2,tn^2}(0) - \ell_{sn^2,tn^2}(-1)| \;\le\; |\ell_{sn^2,tn^2}(0) + \ell_{sn^2,tn^2}(-1)|$$

and $x \mapsto x^2$ is a monotone function of the modulus of x, we see that it is sufficient to estimate the second moment of the sum of the respective local times between sn^2 and tn^2. However, by the coupling just mentioned it is sufficient to estimate

$$\frac{1}{n^2}\mathbb{E}_x\Big[(\xi_{sn^2,tn^2}(0)+\xi_{sn^2,tn^2}(-1))^2\Big]\,,$$

and we obtain the desired estimate as a consequence of Proposition A.3.1. We turn to the analysis of the martingale term. To that end we apply the following version of the Burkholder-Davis-Gundy inequality (see [29], Theorem 4.1):

Theorem 6.2.2. *Let M be a càdlàg square integrable martingale. For any $p>0$ there exists a constant $C=C(p)>0$ such that for all $T>0$,*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|M_t|^p\Big] \;\le\; C\,\mathbb{E}\Big[[M,M]_T^{p/2}\Big]\,.$$

Note that $\mathscr{M}_{tn^2}=\mathscr{M}_{tn^2}(f)-\mathscr{M}_{sn^2}(f)$ is also a martingale in $t\ge s$ whose optional quadratic variation is given by

$$[\mathscr{M},\mathscr{M}]_{tn^2} \;=\; \frac{1}{n^2}\sum_{sn^2\le r\le tn^2}|\Delta_r X^{\text{slow}}|^2\,,$$

where $\Delta_r X^{\text{slow}}$ denotes the size of the jump of X^{slow} at time r. Note that X^{slow} only does jumps of size one, so that the above is $1/n^2$ times the number of jumps in the time interval $[sn^2,tn^2]$. However, since for $\beta\ge 0$ we always have that $\alpha/2n^\beta\le\max\{1/2,\alpha/2\}$, it readily follows that the number of jumps of X^{slow} in the time interval $[sn^2,tn^2]$ is stochastically dominated by the number of jumps of a continuous-time simple symmetric random walk jumping at rate $\max\{1,\alpha\}$, i.e., by $N_{(t-s)n^2}(m)$, where $m=\max\{1,\alpha\}$ and $N(m)$ is a Poisson process with rate m. We can now conclude the proof using that

$$\mathbb{E}\big[N_{(t-s)n^2}(m)\big] \;=\; (t-s)n^2m\,,$$

and this implies that

$$\mathbb{E}[\mathscr{M},\mathscr{M}]_{tn^2} \;\le\; (t-s)m \;<\; \infty,$$

satisfying the required hypothesis.

Chapter 7

Possible directions of extension and difficulties that may be faced

In this chapter we present some direction of extensions of the present work that may be attained in the future, as well as related difficulties that may be faced. In the first section we define the *slow bond Kipnis-Marchioro-Presutti model* (slow bond KMP model) and the conjecture that states this model has a phase transition in the propagation of the local equilibrium. In the second section, we present the slow bond KMP dual process and a result suggesting that the propagation of the local equilibrium is valid for that model. The third section presents a possible coupling technique in order to prove the propagation of local equilibrium, following the outline of [27]. Nevertheless, the slow bond presence represents a difficulty to be dealt with.

7.1 Statements

Inspired by the seminal paper [21], we define the following process. Let $\Omega_{\text{KMP}} := \mathbb{R}_+^{\mathbb{Z}}$ be the state space of reference, whose configurations will be denoted by

$$\theta \;=\; (\ldots,\theta_{-2},\theta_{-1},\theta_0,\theta_1,\theta_2,\ldots)\,,$$

where each θ_i represents an energy on the site i. The Slow Bond KMP Model on $\mathbb{Z}$ is the Markov process $\{\theta(t) : t \geq 0\}$ on the state space whose generator G_N acts on smooth local bounded functions $f \,:\, \Omega_{\text{KMP}} \to \mathbb{R}$ as

$$\begin{aligned}(G_N f)(\theta) \;:=\; &\frac{\alpha}{N^\beta}\int_0^1 \Big[f(\ldots,p(\theta_{-1}+\theta_0),(1-p)(\theta_{-1}+\theta_0),\ldots) - f(\theta)\Big]\,dp\\ &+\sum_{x\in\mathbb{Z}\setminus\{-1\}}\int_0^1 \Big[f(\ldots,p(\theta_x+\theta_{x+1}),(1-p)(\theta_x+\theta_{x+1}),\ldots) - f(\theta)\Big]\,dp\,.\end{aligned}$$

The vector $\theta \;:=\; (\ldots,\theta_{-1},\theta_0,\theta_1,\ldots) \in \Omega_{\text{KMP}}$ will be called an energy configuration, where Ω_{KMP} is the set of energy configurations. This process can be understood as a one-dimensional infinite chain of oscillators, one for each site on $\mathbb{Z}$.

To each edge (pair of neighbour sites) it is associated a Poisson clock of unitary parameter, except for the edge $\{-1, 0\}$, whose Poisson clock has rate α/N^β, where $\alpha > 0$, $\beta \in [0, \infty]$ and $N \in \mathbb{N}$ is the scaling parameter. When a certain clock rings, the total energy at the respective sites is redistributed between them uniformly at random. Then, the $\{-1, 0\}$ edge behaves as a barrier, hence the name *slow bond*.

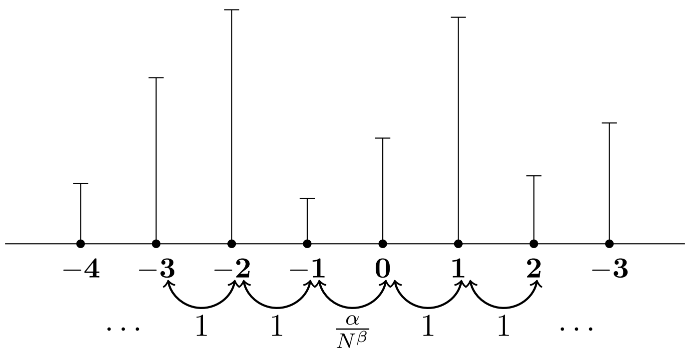

Figure 7.1: A slow bond KMP energy configuration

Considering an infinite volume version of the Kipnis-Marchioro-Presutti process with the presence of a slow bond at the edge $\{-1, 0\}$, we intend to prove the propagation of the local equilibrium, showing a dynamical phase transition according to the slow bond's strength. The conjecture is that if $\beta \in [0, 1)$, the propagation of local equilibrium follows the heat equation; if $\beta = 1$, it follows the heat equation with certain Robin boundary conditions at the origin; and if $\beta > 1$, it follows the heat equation with Neumann boundary conditions at the origin. Now we will provide some definitions that will be useful to state some results and conjectures.

Given $\lambda > 0$, let ν_λ be the Gibbs measure (for the energy) of independent oscillators on $\mathbb{Z}$, which is the following product measure on $\mathbb{R}_+^{\mathbb{Z}}$:

$$d\nu_\lambda = \prod_{x \in \mathbb{Z}} \lambda e^{-\lambda \theta_x} \, d\theta_x \, .$$

It can be checked that this family of measures is invariant, in fact, reversible for the KMP process with a slow bond. The *slowly varying measure* $d\nu^N_{\gamma(\cdot)}$ associated to a profile $\gamma : \mathbb{R} \to \mathbb{R}$ is defined as the product measure

$$d\nu^N_{\gamma(\cdot)} = \prod_{x \in \mathbb{Z}} \lambda^N_x e^{-\lambda^N_x \theta_x} \, d\theta_x \, , \tag{7.1}$$

where $\lambda^N_x = \big(\gamma(\frac{x}{N})\big)^{-1}$. That is, $d\nu^N_{\gamma(\cdot)}$ is a product of exponentials whose parameters λ^N_x depends on the profile γ. Let $\nu_{\gamma(\cdot)} S(tN^2)$ be the distribution of the KMP with a slow bond at time tN^2.

Definition 2. *Let μ be a probability measure on $\mathbb{R}_+$. We say that a sequence of probability measures $d\mu_N$ on $\mathbb{R}_+^{\mathbb{Z}}$ converges locally weakly to a measure $d\mu$ at $u \in \mathbb{R}$ if, for any $k \in \mathbb{N}$, the marginal of μ_N at the sites $\lfloor uN \rfloor - k, \ldots, \lfloor uN \rfloor + k$ converges weakly to the product measure $\bigotimes_{\ell=\lfloor uN \rfloor - k}^{\lfloor uN \rfloor + k} d\mu$.*

In other words, the sequence of measures μ_N converges locally weakly if, around a macroscopic point $u \in \mathbb{R}$, it converges to a product measure with marginal μ, for any finite window around the microscopic point $\lfloor uN \rfloor$.

If the local weak convergence holds for any $u \in \mathbb{R}$, where the measure $d\mu = d\mu(u)$ depends on $u \in \mathbb{R}$, this is known as the *local equilibrium property*, see [20]. If $\gamma : \mathbb{R} \to \mathbb{R}_+$ is bounded and continuous, then the slowly varying measure $d\nu^N_{\gamma(\cdot)}$ has the local equilibrium property, where, for given $u \in \mathbb{R}$, the limiting measure $d\mu(u)$ is an exponential measure of parameter $\gamma(u)^{-1}$.

Consider now some Markov process starting from $d\mu_N$, whose semigroup we denote by $S_N(t)$, that is the distribution of the process is given by $S_N(ta_N)\mu_N$, where a_N is the time-scaling parameter. If the sequence of measures $S_N(ta_N)\mu_N$, under suitable assumptions on μ_N (for instance, assuming that μ_N are slowly varying measures), have the local equilibrium property, we say that the process has the *propagation of local equilibrium property*.

Conjecture 7.1.1. *Let $\gamma : \mathbb{R} \to \mathbb{R}_+$ be a continuous profile, bounded away from zero. Assume that μ_N is a slowly varying sequence of probability measures associated to γ, and let $S_N(tN^2)\mu_N$ be the distribution of the slow bond KMP at time tN^2 started under the measure μ_N.*

Then, $S_N(tN^2)\mu_N$ has the propagation of local equilibrium property. More precisely, for $u \in \mathbb{R}$ and $t > 0$, it converges locally weakly to a product of exponentials of parameter $\rho(t,u)$, where

- *for $\beta \in [0,1)$, ρ is the solution of the heat equation:*

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2}\partial^2_{uu}\rho(t,u), & t \geq 0,\ u \in \mathbb{R} \\ \rho(0,u) = \gamma(u), & u \in \mathbb{R}. \end{cases} \tag{7.2}$$

- *for $\beta = 1$, ρ is the solution of following heat equation with boundary conditions of Robin's type at the origin:*

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2}\partial^2_{uu}\rho(t,u), & t \geq 0,\ u \in \mathbb{R}\backslash\{0\} \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = \alpha\{\rho(t,0^+) - \rho(t,0^-)\}, & t \geq 0 \\ \rho(0,u) = \gamma(u), & u \in \mathbb{R}. \end{cases} \tag{7.3}$$

- *for $\beta \in (1,+\infty]$, ρ is the solution of the heat equation with a boundary condition of Neumann's type at $u = 0$*

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2}\partial^2_{uu}\rho(t,u), & t \geq 0,\ u \in \mathbb{R}\backslash\{0\} \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = 0, & t \geq 0 \\ \rho(0,u) = \gamma(u), & u \in \mathbb{R}. \end{cases} \tag{7.4}$$

In the next two sections we present arguments in favor of the conjecture above.

7.2 Dual process

We construct in this section a discrete process which is dual to the Slow Bond KMP on $\mathbb{Z}$.

Definition 3. *Denote*

$$\eta \;=\; (\ldots,\eta_{-2},\eta_{-1},\eta_0,\eta_1,\eta_2,\ldots)\in\mathbb{N}^{\mathbb{Z}} =: \Omega_{\text{dual}}\,, \tag{7.5}$$

where η_x represents the number of particles at the site $x\in\mathbb{Z}$. Consider the Markov process taking values on Ω_{dual} characterized by the following generator

$$\big(A_N f\big)(\eta) \;=\; \sum_{j\in\mathbb{Z}} \frac{\Big(\mathbb{1}_{[j\neq -1]}+\frac{\alpha}{N^\beta}\mathbb{1}_{[j=-1]}\Big)}{\eta_j+\eta_{j+1}+1}\sum_{q=0}^{\eta_j+\eta_{j+1}}\Big[f(\ldots,\eta_{j-1},q,\eta_j+\eta_{j+1}-q,\ldots)-f(\eta)\Big]\,. \tag{7.6}$$

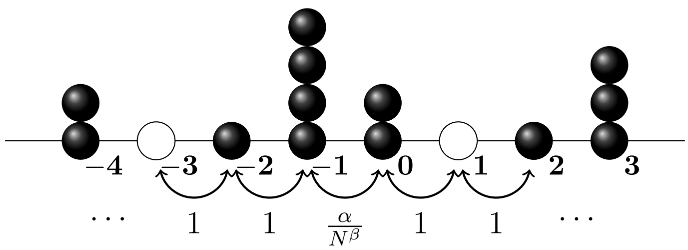

Figure 7.2: Jump rates for the *slow bond random walk for many particles*

This particle system can be described as follows. We associate to each pair of neighbour sites a Poisson clock of parameter one (except the $\{-1,0\}$ edge, whose parameter is $\frac{\alpha}{N^\beta}$). When a Poisson clock rings, the particles in the corresponding sites are uniformly redistributed. Furthermore, all the Poisson clocks are taken as independent. Let $\|\eta\|_1=\sum_{j\in\mathbb{Z}}\eta_j$ be total quantity of particles for a configuration. For $\|\eta\|_1<\infty$ and $\theta\in\Omega_{\text{KMP}}$, we define the duality function $\mathscr{D}:\Omega_{\text{dual}}\times\Omega_{\text{KMP}}\to\mathbb{R}$ by

$$\mathscr{D}(\eta,\theta) \;=\; \prod_{x\in\mathbb{Z}}\frac{\theta_x^{\eta_x}}{\eta_x!}\,. \tag{7.7}$$

Note that the product above is finite indeed due to the assumption $\|\eta\|_1<\infty$. Next, we will make a slight abuse of notation: although already used for the stochastic process, η and θ will design configurations, corresponding to the initial state of the respective Markov processes.

Theorem 7.2.1 (Duality)**.** *Let $\eta\in\Omega_{\text{dual}}$ and $\theta\in\Omega_{\text{KMP}}$. Denote by $\mathbb{E}_\theta$ the expectation induced by the Markov process of generator G_N starting at the configuration θ and denote by $\mathbb{E}_\eta$ the expectation induced by the Markov process of generator A_N starting at the initial configuration η. Then, for all $t\geq 0$,*

$$\mathbb{E}_\eta\Big[\mathscr{D}(\eta(t),\theta)\Big] \;=\; \mathbb{E}_\theta\Big[\mathscr{D}(\eta,\theta(t))\Big]\,. \tag{7.8}$$

The proof of above is very similar to the one in [21, Thm 2.1], and consists on checking that $A_N\mathscr{D}=G_N\mathscr{D}$. Since such an equality is proved by decomposing the generators in parcels associated to *edges*, the slow bond factor here considered makes no difference in the proof. It is possible to check this via [21], disregarding the boundary terms there, which does not exist here. As a consequence of the duality, we obtain:

Proposition 7.2.2. *Assume the hypothesis of Conjecture 7.1.1. In this case, for $u \in \mathbb{R}$ and $t > 0$, we have that*

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\theta(tN^2, \lfloor uN \rfloor)\Big] \;=\; \rho(t,u)\,,$$

where $\rho(t,u)$ is

- *the solution of* (7.2) *if $\beta \in [0,1)$;*
- *the solution of* (7.3) *if $\beta = 1$;*
- *the solution of* (7.4) *if $\beta \in (1, +\infty]$,*

where $\theta(t,x)$ represents the energy quantity at the site $\lfloor uN \rfloor$ in time tN^2 for the process $\theta(t)$.

Proof. Let $\eta = \delta_y$ be the configuration which has a single particle, located at the site $y \in \mathbb{Z}$. It is simple to check that the process starting from η coincides with the *slow bond random walk*. Remember that this model is the Feller process on $\mathbb{Z}$ denoted by $\{X_t^{\text{slow}} : t \geq 0\}$, whose generator L_n acts on local functions $f : \mathbb{Z} \to \mathbb{R}$ via

$$\mathsf{L}_N f(x) \;=\; \tau^N_{x,x+1}\Big[f(x+1) - f(x)\Big] + \tau^N_{x,x-1}\Big[f(x-1) - f(x)\Big]\,, \tag{7.9}$$

where

$$\tau^N_{x,x+1} \;=\; \tau^N_{x+1,x} \;=\; \begin{cases} \dfrac{\alpha}{2N^\beta}, & \text{if } x = -1,\\ 1/2, & \text{otherwise.}\end{cases} \tag{7.10}$$

Note that the process of generator A_N as defined in (7.6) starting with a single particle is a *lazy* random walk. That is, when the Poisson associated to a given edge rings, the particle stands still with probability one half, which implies the $1/2$ factor in the rates (7.10) above.

By the equations (7.8) and (7.7), we obtain, respectively, the equalities below

$$\mathbb{E}_\theta\big[\mathscr{D}(\eta, \theta(t))\big] \;=\; \mathbb{E}_\eta\big[\mathscr{D}(\eta(t), \theta)\big] \;=\; \mathbb{E}_\eta\left[\prod_{x\in\mathbb{Z}} \frac{\theta_x^{\eta(t,x)}}{\eta(t,x)!}\right],$$

where $\eta(t,x)$ represents the quantity of particles at the site x in time t for the process $\eta(t)$. Since this process is conservative and starts from $\eta = \delta_x$, we have that $\eta(t,x) \in \{0,1\}$ for any positive time. Denote $f : \mathbb{Z} \to \mathbb{R}$ as $f(x) = \theta_x$, where θ_x is the initial configuration for the KMP process with a slow bond. Since $\eta(t,x) \in \{0,1\}$ and the fact that the process $\eta(t,x)$ starting from a configuration with a single particle is the slow bond random walk, then

$$\mathbb{E}_\theta\big[\mathscr{D}(\delta_x, \theta(t))\big] \;=\; \mathbb{E}\big[\theta(t,x)\big] \;=\; \mathbb{E}_x\big[f(X_t^{\text{slow}})\big]\,,$$

where $\mathbb{E}_x$ denotes the expectation with respect to the probability $\mathbb{P}_x$, which represents the probability associated to the slow bond random walk started at site x. In particular, letting $x = \lfloor uN \rfloor$ and choosing tN^2 as the time, we obtain that

$$\mathbb{E}_\theta\big[\mathscr{D}(\delta_{\lfloor uN \rfloor}, \theta(tN^2))\big] \;=\; \mathbb{E}_{\lfloor uN \rfloor}\big[f(X^{\text{slow}}_{tN^2})\big]\,. \tag{7.11}$$

Define now

$$\theta_j^N \;=\; \gamma\Big(\frac{j}{N}\Big)\,, \quad \forall j\in\mathbb{Z}\,,$$

where $\gamma:\mathbb{R}\to\mathbb{R}_+$ is the bounded continuous profile in the Theorem 7.1.1. Hence, replacing the initial condition θ in (7.11) by θ^N, we get

$$\mathbb{E}_{\theta^N}\big[\mathscr{D}(\delta_{\lfloor uN\rfloor},\theta(t))\big] \;=\; \mathbb{E}_{\lfloor uN\rfloor}\bigg[\gamma\bigg(\frac{X^{\text{slow}}_{tN^2}}{N}\bigg)\bigg]\,.$$

By Theorem 5.6.1 and (7.11), we have that $\mathbb{E}_{\theta^N}\big[\mathscr{D}(\delta_{\lfloor uN\rfloor},\theta(t))\big]$ converges to

(a) $\mathbb{E}_u\big[\gamma(B_t)\big]$, if $\beta\in[0,1)$;

(b) $\mathbb{E}_u\big[\gamma(B_t^{\text{snob}})\big]$, if $\beta=1$;

(c) $\mathbb{E}_u\big[\gamma(\text{sgn}(u)|B_t|)\big]$, if $\beta\in(1,+\infty]$.

It is known that $\mathbb{E}_u\big[\gamma(B_t)\big]$ and $\mathbb{E}_u\big[\gamma(\text{sgn}(u)|B_t|)\big]$ are, respectively, the solutions of PDE's (7.2) and (7.4). Furthermore, it was proved in Section 3.1 that $\mathbb{E}_u\big[\gamma(B_t^{\text{snob}})\big]$ is the solution of (7.3). □

An important role can be played by the so-called $\mathscr{V}$ *correlation function*, in order to prove the propagation of local equilibrium, which is given by

$$\mathscr{V}(\theta,x_1,\dots,x_n;t) \;=\; \mathbb{E}_\theta\mathscr{D}\bigg(\sum_{i=1}^n\delta_{x_i},\theta(t)\bigg)-\prod_{i=1}^n\mathbb{E}_\theta\mathscr{D}(\delta_{x_i},\theta(t)). \tag{7.12}$$

Here $\mathbb{E}_\theta$ denotes the expectation induced by the Markov process of generator G_N starting at the configuration θ, as defined in Theorem 7.2.1, and $\sum_{i=1}^n\delta_{x_i}$ denotes the configuration with particles located at positions $x_1,\dots,x_n$.

The main aim is to prove that $\mathscr{V}$ goes to zero when t goes to infinity, as done in [10] for the symmetric exclusion process (SEP) and in [27] for the symmetric inclusion process (SIP). However, in these works it was strongly used that the SEP and the SIP are self-dual process, which is not valid for the KMP process.

7.3 Coupling

Here, we present a theorem from [27], which shows the propagation of local equilibrium of the symmetric inclusion process (SIP), see [27] for a definition and more details. First, consider the coupling $(X(t),Y(t),U(t),V(t))$, such that $(X(t),Y(t))$ evolves as two SIP(m) particles, and $U(t)$ and $V(t)$ are two independent random walkers moving at rate $\frac{m}{2}$.

In order to define the generator of this coupling, we denote by e_{13} the vector $(1,0,1,0)$ and e_{24} the vector $(0,1,0,1)$. Moreover, defining $\text{x}:=(x,y,u,v)$, we have that the coupling generator is

$$
\begin{aligned}
\mathscr{L} f(\mathbf{x}) \;=\; & \frac{m}{2}\sum_{\epsilon=\pm 1}\left[\big(f(\mathbf{x}+\epsilon e_{13})-f(\mathbf{x})\big)+\big(f(\mathbf{x}+\epsilon e_{24})-f(\mathbf{x})\big)\right] \\
+\; & \mathbb{1}_{\{|x-y|=1\}}\big(f(x,x,u,v)+f(y,y,u,v)-2f(x,y,u,v))\big).
\end{aligned}
$$

This coupling is interpreted in this way: random walk jumps are performed together, while the inclusion jumps are performed only in the first two coordinates. The follow result, which is an ingredient to proof the propagation of local equilibrium, claims that, when the particles begin at the same site, we can keep them closer than $\rho\sqrt{t}$ for t large, for all $\rho > 0$, with probability close to one.

Theorem 7.3.1 ([27]). *In the basic coupling $(X(t), Y(t), U(t), V(t))$ between two SIP(m) and two independent random walkers moving at rate $\frac{m}{2}$, starting at the same initial positions, we have*

$$
\lim_{t\to\infty}\frac{|X(t)-U(t)|^2}{t}=0, \tag{7.13}
$$

where the limit is in L^1, and hence in probability, for every position with $X(0) = U(0), Y(0) = V(0)$. The same statement holds for $|Y(t) - V(t)|$.

In order to obtain propagation of the local equilibrium for the KMP, we may need to prove a version of Theorem 7.3.1 for the slow bond random walk. Our version of this theorem consider $(X(t), Y(t))$ evolving as two particles in slow bond RW and $U(t), V(t)$ as two independent particles in the same process. Nevertheless, the slow bond presence makes this a difficult task .

Another possibility is to prove that the limit in (7.13) for the symmetric random walk implies the same limit for the slow bond RW. However, theoretically the slow bond makes the particles in $X(t)$ and $U(t)$ to be closer than in symmetric case.

Appendix A

Auxiliary tools

A.1 An explicit solution to a PDE with Robin boundary conditions

Here we summarize the idea from [12] on how to obtain the explicit solution of PDE (2.4).

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho, \quad u \neq 0 \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = \frac{\kappa}{2}[\rho(t,0^+) - \rho(t,0^-)] \\ \rho(0,u) = f(u). \end{cases} \tag{A.1}$$

Denote by $T_t^\kappa f(u)$ the solution of (A.1), where f is the initial condition and denote by $f_{\text{even}}(u)$ and $f_{\text{odd}}(u)$ its even and odd parts, respectively. By linearity, the solution may be written as the sum of $T_t^\kappa f_{\text{even}}(u)$ and $T_t^\kappa f_{\text{odd}}(u)$. Since the PDE (A.1) preserves parity, we conclude that $T_t^\kappa f_{\text{even}}(u)$ is solution of

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho \\ \rho(0,u) = f_{\text{even}(n)}(u) \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = 0, \end{cases} \tag{A.2}$$

which boundary condition can be dropped due to the fact that f_{even} is an even function. That is, $T_t^\kappa f_{\text{even}}(u)$ is simply the solution of usual heat equation

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho \\ \rho(0,u) = f_{\text{even}}(u). \end{cases} \tag{A.3}$$

which solution is given by the classical formula

$$\rho(t,u) \;=\; \int_{\mathbb{R}} f_{\text{even}}(u-y)\frac{e^{-y^2/2t}}{\sqrt{2\pi t}}dy\,.$$

On the other hand, again by preservation of parity, we can deduce that $T_t^\kappa f_{\text{odd}}(u)$ is given by

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho, \quad u > 0 \\ \rho(0,u) = f_{\text{odd}}(u), \quad u > 0 \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = \kappa\rho(t,0^+), \quad t > 0. \end{cases} \tag{A.4}$$

on the positive half line, with analogous definition on the negative half line. The standard technique to solve the (A.4) is to define

$$v(t,u) := \kappa\rho(t,u) - \partial_u \rho(t,u)\,, \tag{A.5}$$

which will be solution of

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v, \quad u>0 \\ v(0,u) = \kappa f_{\text{odd}}(u) - f'_{\text{odd}}(u) \;\; u>0 \\ v(t,0) = 0. \end{cases} \tag{A.6}$$

with analogous definition for the negative half line. Note that the equation above has Dirichlet boundary conditions, which can easily solved by the *image method* (see [18], p. 56). Once we have the expression for v, solving the linear ODE (A.5) gives us the expression for $T_t^\kappa f_{\text{odd}}(u)$.

A.2 Scale invariance

Proposition A.2.1. *The random variables $L(x,t)$ and $L(x\sqrt{n},tn)/\sqrt{n}$ have the same distribution.*

Proof. Doing the changing of variables $u = sn$, we get

$$\begin{aligned} L(x,t) &= \lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{B_s\in(x-\varepsilon,x+\varepsilon)}ds = \lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon n}\int_0^{tn} \mathbb{1}_{B_{u/n}\in(x-\varepsilon,x+\varepsilon)}\frac{du}{n} \\ &= \lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon n}\int_0^{tn} \mathbb{1}_{\sqrt{n}B_{u/n}\in(x\sqrt{n}-\varepsilon\sqrt{n},x\sqrt{n}+\varepsilon\sqrt{n})}\frac{du}{n}\,. \end{aligned}$$

Due to the BM's scaling invariance, the last expression is equal in law to

$$\lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon n}\int_0^{tn} \mathbb{1}_{B_u\in(x\sqrt{n}-\varepsilon\sqrt{n},x\sqrt{n}+\varepsilon\sqrt{n})}\frac{du}{n} = \frac{L(x\sqrt{n},tn)}{\sqrt{n}}\,.$$

□

A.3 L^p norm estimates for local times

The next result is likely standard, however we were not able to find it in the literature, so we provide a proof. Recall that $\xi_{tn^2}(0)$ denotes the local time of simple random walk at the origin.

Proposition A.3.1. *Let $q\in\mathbb{N}$, then for all $s,t>0$ there exists a constant $C>0$ such that for all $n\in\mathbb{N}$ and all $x\in\mathbb{Z}$,*

$$\mathbb{E}_x\left[(\xi_{sn^2,tn^2}(0))^q\right] \lesssim |t-s|^{\frac{q}{2}}n^q\,.$$

Proof. For simplicity we prove the result only for $s = 0$, however since our estimates are uniform in the starting point, the general case is a straightforward consequence. First note that a change of variables yields that

$$\xi_{tn^2}(0) \;=\; \int_0^{tn^2} \mathbb{1}_{\{X_s=0\}}\, ds \;=\; n^2 \int_0^t \mathbb{1}_{\{X_{sn^2}=0\}}\, ds\,.$$

We then see that

$$\mathbb{E}\big[(\xi_{tn^2}(0))^q\big] \;=\; n^{2q} q! \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{q-1}}^t ds_q \prod_{i=1}^q p_{(s_i - s_{i-1})n^2}(0)\,, \qquad \text{(A.7)}$$

where we set $s_0 = 0$. We apply now the local central limit theorem [22, Theorem 2.5.6], which states that there is a constant c such that for all t and all n

$$n p_{tn^2}(0) \;\leq\; \frac{c}{\sqrt{t}}\,.$$

Plugging this estimate into (A.7) we may now finish the proof. □

A.4 Proof of Lemma 5.2.1

Next we furnish a short proof of Lemma 5.2.1.

Proof of Lemma 5.2.1. By the triangle inequality, we have

$$\begin{aligned}\Big|\mathbf{P}_t^n f_{\text{even}(n)}\big(\tfrac{\lfloor un\rfloor}{n}\big) - P_t f_{\text{even}}(u)\Big| \;\leq\; & \Big|\mathbf{P}_t^n f_{\text{even}(n)}\big(\tfrac{\lfloor un\rfloor}{n}\big) - P_t f_{\text{even}}\big(\tfrac{\lfloor un\rfloor}{n}\big)\Big| \\ & + \Big|P_t f_{\text{even}}(u) - P_t f_{\text{even}}\big(\tfrac{\lfloor un\rfloor}{n}\big)\Big|.\end{aligned}$$

Due to the Lipschitz continuity of $u \mapsto P_t f_{\text{even}}(u)$, the last parcel above is bounded by a term of order $\frac{1}{n}$. Then, it is enough to prove (5.11) with $P_t f_{\text{even}}(u)$ replaced by $P_t f_{\text{even}(n)}\big(\frac{\lfloor un\rfloor}{n}\big)$. Moreover, with a slight abuse of notation un will henceforth denote the integer part of un. Then, using the equation (5.8) and symmetry $p_t(x) = \mathbb{P}_0\big[X_t = x\big] = \mathbb{P}_x\big[X_t = 0\big]$, we can now write

$$\mathbf{P}_t^n f_{\text{even}(n)}\big(\tfrac{\lfloor un\rfloor}{n}\big) \;=\; \sum_{z \in \frac{1}{n}\mathbb{Z}} f_{\text{even}(n)}(z) p_{tn^2}(n(u-z))\,. \qquad \text{(A.8)}$$

We now apply the local central limit theorem, [22, Theorem 2.3.11] which states that for $x \in \frac{1}{n}\mathbb{Z}$,

$$n p_{tn^2}(nx) \;=\; K_t(x) \exp\left\{ O\Big(\frac{1}{tn^2} + \frac{|nx|^4}{(tn^2)^3}\Big)\right\},$$

where $K_t(x)$ is the usual heat kernel, as defined in (4.10). We use this estimate in (A.8) for all $z \in \frac{1}{n}\mathbb{Z}$ such that $|n(u-z)| \leq n^{5/4}$. Since there exists a constant $C > 0$ such that for all $x \in [0,1)$ we have the estimate $|e^x - 1| \leq C|x|$ the above states that

$|np_{tn^2}(n(u-z)) - K_t(u-z)| \leq \frac{C}{n}$ for the range of z's just mentioned. Moreover, note that

$$\Big| \sum_{\substack{z\in\frac{1}{n}\mathbb{Z}: \\ |n(u-z)|\geq n^{5/4}}} f_{\text{even}(n)}(z) p_{tn^2}(n(u-z)) \Big| \; \leq \; \|f\|_\infty \, \mathbb{P}_0\big[\, |X_{tn^2}| \geq n^{5/4} \,\big] \, ,$$

and by [22, Proposition 2.1.2 (b)] we see that the above is bounded by $C_1 e^{-C_2 n^{1/8}}$, for some constants C_1 and C_2. The proof may now be finished by using the above approximation of the continuous heat kernel by the discrete one and by a standard Riemann sum approximation. We omit the details. □

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