Análise no $\mathbb{R}^n$ MAT 508 2012.2

Prova III

Data: 28/11/12

Duração: 2h

GABARITO

1:) Como w é k-forma diferencial em $\mathbb{R}^n$

$$w = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$\Rightarrow dw = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{l=1}^{n} \frac{\partial a_{i_1 \cdots i_k}}{\partial x_l} dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$\Rightarrow d(dw) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial^2 a_{i_1 \cdots i_k}}{\partial x_l \partial x_m} dx^m \wedge dx^l \wedge dx^{i_1} \wedge \cdots$$

$$\cdots \wedge dx^{i_k} =$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left[\sum_{\substack{l=m \\ l=1,\dots,n}} \frac{\partial^2 a_{i_1 \cdots i_k}}{\partial x_l \partial x_l} \underbrace{dx^l \wedge dx^l}_{=0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right.$$

$$+ \sum_{l \neq m} \underbrace{\left(\frac{\partial^2 a_{i_1 \cdots i_k}}{\partial x_l \partial x_m} dx^l \wedge dx^m + \frac{\partial^2 a_{i_1 \cdots i_k}}{\partial x_m \partial x_l} dx^m \wedge dx^l \right)}_{=0 \text{ por Schwarz e } dx^l \wedge dx^m = -dx^m \wedge dx^l.} \wedge$$

$$\wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

2:) Por definição, $v_1 \times \cdots \times v_{n-1}$ é o vetor em $\mathbb{R}^n$ tal que, $\forall\, u \in \mathbb{R}^n$,

$$\langle u, v_1 \times \cdots \times v_{n-1} \rangle = \det(v_1, \ldots, v_{n-1}, u)$$

Escolha $u = v_i$. Como det é zero se duas entradas são iguais, isto implica

$$\langle v_i, v_1 \times \cdots \times v_{n-1} \rangle = 0.$$

3:) Defina $\det(Bv_1, \ldots, Bv_n)$, onde $v_1, \ldots, v_n \in \mathbb{R}^n$ são vetores coluna. Esta função é multilinear e alternada em $v_1, \ldots, v_n$. Como $\dim \Omega^n(\mathbb{R}^n) = 1$,

$$\det(Bv_1, \ldots, Bv_n) = \lambda \det(v_1, \ldots, v_n) \quad (*)$$

Escolha $v_1 = e_1, \ldots, v_n = e_n$. Daí, concluímos que $\lambda = \det(B)$. Sejam $u_1, \ldots, u_n \in \mathbb{R}^n$ as colunas da matriz A. Substituindo em $(*)$,

$$\det(AB) = \det(Bu_1, \ldots, Bu_n) = \lambda \det(u_1, \ldots, u_n)$$

$$\Rightarrow \det(AB) = \det(B)\det(A).$$

4:) $d(\alpha\,dx + \beta\,dy) =$

$$= \left(\frac{\partial\alpha}{\partial x}dx + \frac{\partial\alpha}{\partial y}dy\right)\wedge dx + \left(\frac{\partial\beta}{\partial x}dx + \frac{\partial\beta}{\partial y}dy\right)\wedge dy$$

$$= \frac{\partial\alpha}{\partial y}dy\wedge dx + \frac{\partial\beta}{\partial x}dx\wedge dy$$

$$= \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right)dx\wedge dy$$