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Non-equilibrium and stationary fluctuations of a slowed boundary symmetric exclusion

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Abstract

We consider a one-dimensional symmetric simple exclusion process in contact with slowed reservoirs: at the left (resp. right) boundary, particles are either created or removed at rates given by α/n or $(1-\alpha)/n$ (resp. β/n or $(1-\beta)/n$) where $\alpha, \beta > 0$ and n is a scaling parameter. We obtain the non-equilibrium fluctuations and from the latter we obtain also the non-equilibrium stationary fluctuations.
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1. Introduction

One of the most difficult problems in the field of interacting particle systems is the rigorous mathematical derivation of the non-equilibrium fluctuations of a system around its hydrodynamic limit. The main difficulty one faces when trying to show that result is the fact that the systems exhibit long-range space–time correlations and for that reason the non-equilibrium fluctuations have only been derived for very few models, see, for example, [4,13,17] and references therein.

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Moreover, the study of non-equilibrium steady states has attracted a lot of attention over the last twenty years and up to now the microscopic description of these states is still incipient, see, for example, the review [3].

In [15], the non-equilibrium stationary fluctuations for the symmetric simple exclusion in contact with fixed reservoirs were derived as a consequence of its non-equilibrium fluctuations. In this article we examine the dynamical non-equilibrium fluctuations of the symmetric simple exclusion process in contact with *slowed* reservoirs. In this model the exclusion dynamics is superposed with a Glauber dynamics at each endpoint of a one-dimensional lattice with $n-1$ points. According to this dynamics, particles perform continuous time symmetric random walks in the discrete lattice $\{1,\dots,n-1\}$ which we call bulk, in such a way that two particles cannot occupy the same site at a given time, the so-called exclusion rule, and at the endpoints of the bulk, namely at the sites 1 and $n-1$, particles can be injected and removed at a certain rate, which is slowed with respect to the jump rate in the bulk.

Our main interest is the derivation of the non-equilibrium fluctuations and the non-equilibrium stationary fluctuations for this model. We chose a regime in which the Glauber dynamics is slowed enough so that the hydrodynamic behavior of the system is macroscopically different from the case in which the Glauber dynamics is not slowed, as in [15], for instance. More precisely, in [15] the Glauber dynamics is defined in such a way that particles can get in and out of the system at rate α and β, respectively. In our model, these rates are slowed by a factor depending on a parameter n. As a consequence of having slowed reservoirs, the hydrodynamical profile in our model is different from the one of [15], the latter being a solution of the heat equation with Dirichlet boundary conditions in which the solution is fixed at the boundaries by $\rho(t,0)=\alpha$ and $\rho(t,1)=\beta$. In the model considered here, it has been proved in [1] that the hydrodynamical profile is a solution of the heat equation with a type of Robin boundary conditions in which the value of the profile at the boundaries is not fixed, but instead it fixes the values of its spatial derivative, namely: $\partial_u\rho(t,0)=\rho(t,0)-\alpha$ and $\partial_u\rho(t,1)=\beta-\rho(t,1)$, see (2.4). These boundary conditions reflect the fact that the mass transfer, given by $\partial_u\rho(t,\cdot)$, at the boundaries is proportional to the difference of concentration. Contrarily to what happens in the model of [15] which fixes the density at the reservoirs, in our case we do not have $\rho(t,0)=\alpha$, so that the term $\rho(t,0)-\alpha$ represents the difference of concentration between the bulk and the boundary. We also note that in [1] it has been analyzed the hydrodynamic limit for a generalization of our model. There, the rates at the reservoirs are slowed with respect to the rate in the bulk by a factor n^θ, where $\theta>0$, and our model corresponds to the choice $\theta=1$. We note that, as proved in [1], for $\theta<1$ (resp. $\theta>1$) the hydrodynamical profile is the unique weak solution of the heat equation with Dirichlet (resp. Neumann) boundary conditions.

We would also like to refer other articles on this subject as, for example, [5–7], where the authors consider models with slowed boundaries but one boundary acts only for the creation of particles and the other boundary acts only on the annihilation of particles. As a consequence, the density of particles in the reservoirs remains the same, and the hydrodynamical profile in such case is a solution of the heat equation with Dirichlet boundary conditions.

We observe that when $\alpha=\beta=\rho$, the reservoirs do not induce any current in the system contrarily to what happens if, for example, $\alpha<\beta$, since in this case particles can get in the system more easily from the right boundary, and there is a current of particles, due to the reservoirs, from the right reservoir to the left reservoir. In the case $\alpha=\beta=\rho$, the Bernoulli product measures given by $\nu_\rho\{\eta:\eta(x)=1\}=\rho$ are invariant and due to the absence of an external current, they are called equilibrium measures. However, in the non-equilibrium scenario, that is when $\alpha \neq \beta$, this fact is no longer true. Nevertheless, there exists a unique stationary measure that we denote by μ_{ss}. Since $\alpha \neq \beta$, the reservoirs induce a current of particles in the system and for that reason μ_{ss} is a non-equilibrium stationary measure. This measure has been partially characterized in, for example, [8] and it has been proved in [1, Theorem 2.2] that it is associated to a profile $\overline{\rho}(\cdot)$ which is stationary with respect to the hydrodynamic equation, so that $\overline{\rho}(\cdot)$ is linear and $\overline{\rho}(0) = \frac{2\alpha+\beta}{3}$ and $\overline{\rho}(1) = \frac{\alpha+2\beta}{3}$. We emphasize here that, as one can see from the previous properties on the stationary profile, in our model the density at the reservoirs is not fixed as being α at $u = 0$ and β at $u = 1$.

To analyze the non-equilibrium fluctuations we consider a space of smooth test functions f satisfying the boundary conditions of the homogeneous hydrodynamic equation, that is, the hydrodynamic equation with $\alpha = \beta = 0$. Our setting for initial states is quite general and can be described as follows. We consider initial measures μ_n associated to a measurable profile $\rho_0 : [0, 1] \to [0, 1]$ in the sense of (2.3). Moreover, denoting for $x \in \{1, \dots, n-1\}$, $\rho_0^n(x) = \mathbb{E}_\mu[\eta(x)]$, we ask $\rho_0^n(\cdot)$ to be close to the given $\rho_0(\cdot)$ as stated in Assumption 2.2 and we also ask that the corresponding space correlations to vanish as $n \to \infty$, as stated in Assumption 2.3. In this case we show that the sequence of density fluctuation fields is tight and we characterize its limiting points so that, for a fixed time t, the solution is given by the sum of a Gaussian random variable and the initial condition, see the relation (2.14). Besides that, if on top of the aforementioned assumptions we ask that at the initial time the sequence of density fields converges to a mean-zero Gaussian process, then the convergence takes place and the limiting process is an Ornstein–Uhlenbeck process solution of (2.18). We also note that from our results we can obtain the non-equilibrium fluctuations starting from a product measure with slowly varying density. More precisely, if we fix a profile $\gamma : [0, 1] \to [0, 1]$ and consider μ_n as the Bernoulli product measure such that $\mu_n\{\eta : \eta(x) = 1\} = \gamma(\frac{x}{n})$, then the result also holds, leading to an Ornstein–Uhlenbeck process in the limit.

As a consequence of the previous results we can derive the non-equilibrium stationary fluctuations. For that purpose we just have to check that the imposed conditions on the initial states are satisfied by the non-equilibrium stationary state and to recover the corresponding covariance we perform a careful analysis of the time limit of the covariance obtained in the general non-equilibrium scenario.

To prove the non-equilibrium fluctuations, since we consider the system starting from general initial measures, which can develop long-range correlations, we need a sharp bound on the space correlations in order to make our method work. For that purpose we make a careful analysis of solutions of a bidimensional discrete scheme which has non-trivial boundary conditions.

As a future work we plan to derive our results for the models studied in [1] for the case $\theta \neq 1$. The main difficulty we will face is the derivation of sharp bounds on the space correlations of the system, and we will also need to perform a careful analysis of some additive functionals associated to the system.

Here follows an outline of this article. In Section 2 we present the model, we recall its hydrodynamic limits and we enunciate our results, namely: Theorem 2.4, where we state the non-equilibrium fluctuations for general initial measures; Theorem 2.5, where we state the non-equilibrium fluctuations when the limit is an Ornstein–Uhlenbeck process for which the initial measures have to satisfy a Gaussian central limit theorem and, as a consequence of the previous results; Theorem 2.8 where we state the non-equilibrium stationary fluctuations. In Section 3 we present some necessary results related to the hydrodynamic equation and its semigroup. In Sections 4, 5 and 6 we prove, respectively, Theorems 2.5, 2.4 and 2.8. Section 7 is devoted to tightness and Section 8 is devoted to space correlations estimates.

2. Statement of results

2.1. The model

Given $n \geq 1$ let $\Sigma_n = \{1, \dots, n-1\}$. The symmetric simple exclusion process with slow boundaries is a family of Markov processes $\{\eta_t : t \geq 0\}$ with state space $\Omega_n := \{0,1\}^{\Sigma_n}$. We denote the configurations of the state space Ω_n by η, so that for $x \in \Sigma_n$, $\eta(x) = 0$ means that the site x is vacant while $\eta(x) = 1$ means that the site x is occupied. We characterize this Markov process in terms of its infinitesimal generator $\mathscr{L}_n$ as follows. Let $\mathscr{L}_n = \mathscr{L}_{n,o} + \mathscr{L}_{n,b}$, where, for a given function $f : \Omega_n \to \mathbb{R}$, we have

$$(\mathscr{L}_{n,o} f)(\eta) = \sum_{x=1}^{n-2} \Big(f(\eta^{x,x+1}) - f(\eta) \Big), \tag{2.1}$$

$$(\mathscr{L}_{n,b} f)(\eta) = \frac{1}{n} \sum_{x \in \{1, n-1\}} \Big[r_x(1-\eta(x)) + (1-r_x)\eta(x) \Big] \Big(f(\sigma^x \eta) - f(\eta) \Big), \tag{2.2}$$

with $r_1 = \alpha$ and $r_{n-1} = \beta$. Above, for $x \in \{1, \dots, n-2\}$, the configuration $\eta^{x,x+1}$ is obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, i.e.,

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x+1, \\ \eta(y), & \text{otherwise,} \end{cases}$$

and for $x \in \{1, n-1\}$ the configuration $\sigma^x \eta$ is obtained from η by flipping the occupation variable $\eta(x)$, i.e,

$$(\sigma^x \eta)(y) = \begin{cases} 1 - \eta(y), & \text{if } y = x, \\ \eta(y), & \text{otherwise.} \end{cases}$$

The dynamics of this model can be described in words in the following way. In the bulk, particles move accordingly to continuous time symmetric random walks under the additional exclusion rule: whenever a particle tries to jump to an occupied site, such jump is suppressed. Additionally, at the left boundary, particles can be created (resp. removed) at rate α/n (resp. at rate $(1-\alpha)/n$) and at the right boundary, particles can be created (resp. removed) at rate β/n (resp. at rate $(1-\beta)/n$). See Fig. 1 for an illustration. Note that when $\alpha = \beta = \rho$, for which there is no external current induced by the reservoirs, it is easy to check that the Bernoulli product measures given by $\nu_\rho\{\eta : \eta(x) = 1\} = \rho$ are invariant. However, when $\alpha \neq \beta$ this is no longer true. Nevertheless, for $\alpha \neq \beta$, there is a unique stationary measure of the system, that we denote by μ_{ss}, which is no longer a product measure. For further properties on this measure we refer the reader to, for example, [8]. In particular, it is shown in [1, Theorem 2.2] that this measure is associated to a profile $\overline{\rho}(\cdot)$ which is stationary with respect to the hydrodynamic equation, so that $\overline{\rho}(\cdot)$ is linear and $\overline{\rho}(0) = \alpha + \frac{\beta-\alpha}{3}$ and $\overline{\rho}(1) = \alpha + 2\frac{\beta-\alpha}{3}$. We observe that in the case where the reservoirs are not slowed, as in [15], the stationary profile associated to the hydrodynamic equation, which is the heat equation with Dirichlet boundary conditions, is the linear interpolation between α and β.

2.2. Hydrodynamic limit

Fix a measurable density profile $\rho_0 : [0,1] \to [0,1]$. For each $n \in \mathbb{N}$, let μ_n be a probability measure on Ω_n. We say that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is *associated* to the profile $\rho_0(\cdot)$ if, for any

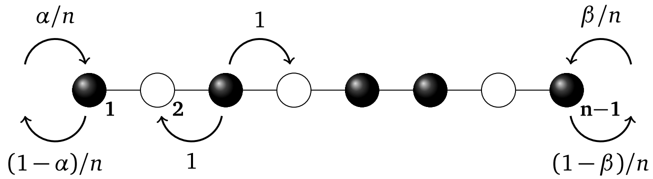

Fig. 1. Illustration of jump rates. The leftmost and rightmost rates are the entrance/exiting rates.

$\delta > 0$ and any continuous function $f : [0, 1] \to \mathbb{R}$ the following limit holds:

$$\lim_{n\to\infty} \mu_n\Big[\eta : \Big|\frac{1}{n}\sum_{x=1}^{n-1} f(\tfrac{x}{n})\,\eta(x) - \int f(u)\,\rho_0(u)\,du\Big| > \delta\Big] = 0\,. \tag{2.3}$$

Fix $T > 0$. Let $\mathscr{D}([0, T], \Omega_n)$ be the space of trajectories which are right continuous, with left limits and taking values in Ω_n. Denote by $\mathbb{P}_{\mu_n}$ the probability on $\mathscr{D}([0, T], \Omega_n)$ induced by the Markov process with generator $n^2\mathscr{L}_n$ and the initial measure μ_n and denote by $\mathbb{E}_{\mu_n}$ the expectation with respect to $\mathbb{P}_{\mu_n}$. From [1] we have the following result, known in the literature as *hydrodynamic limit*.

Theorem 2.1 (*Hydrodynamic Limit, [1]*). *Suppose that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is associated to a profile $\rho_0(\cdot)$ in the sense of (2.3). Then, for each $t \in [0, T]$, for any $\delta > 0$ and any continuous function $f : [0, 1] \to \mathbb{R}$,*

$$\lim_{n\to+\infty} \mathbb{P}_{\mu_n}\Big[\eta_\cdot : \Big|\frac{1}{n}\sum_{x=1}^{n-1} f(\tfrac{x}{n})\,\eta_{tn^2}(x) - \int f(u)\,\rho(t, u)\,du\Big| > \delta\Big] = 0\,,$$

where $\rho(t, \cdot)$ is the unique weak solution of the heat equation with Robin boundary conditions given by

$$\begin{cases} \partial_t\rho(t, u) = \partial_u^2\rho(t, u)\,, & \text{for } t > 0\,,\ u \in (0, 1)\,,\\ \partial_u\rho(t, 0) = \rho(t, 0) - \alpha\,, & \text{for } t > 0\,,\\ \partial_u\rho(t, 1) = \beta - \rho(t, 1)\,, & \text{for } t > 0\,,\\ \rho(0, u) = \rho_0(u)\,, & u \in [0, 1]\,. \end{cases} \tag{2.4}$$

2.3. Density fluctuations

2.3.1. The space of test functions

By $f \in C^\infty([0, 1])$ we mean that both $f : [0, 1] \to \mathbb{R}$ as well as all its derivatives are continuous functions in $[0, 1]$. Next, we define a subspace of $C^\infty([0, 1])$ which is intrinsically associated to the limiting fluctuations, as we shall see later on. Inspired in [10,9], we define

Definition 2.1. Let $\mathscr{S}$ denote the set of functions $f \in C^\infty([0, 1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that $\partial_u^{2k+1} f(0) = \partial_u^{2k} f(0)$ and $\partial_u^{2k+1} f(1) = -\partial_u^{2k} f(1)$.

Notice that for $k = 0$, the conditions above are nothing but the boundary conditions that appear in the homogeneous version of (2.4), i.e., imposing $\alpha = \beta = 0$. For $k = 1$, the conditions above are again these boundary conditions, but imposed for the Laplacian of f, and so on.

Definition 2.2. Let $T_t : \mathscr{S} \to \mathscr{S}$ be the semigroup associated to (2.4) with $\alpha = \beta = 0$. That is, given $f \in \mathscr{S}$, by $T_t f$ we mean the solution of the homogeneous version of (2.4) with initial condition f.

Rigorously speaking, above we should not have written $T_t : \mathscr{S} \to \mathscr{S}$, since we do not know yet if the image of T_t is contained in $\mathscr{S}$. But this is true and it will be proved below in Corollary 3.2.

Remark 2.2. Although not used in what follows, we observe that the solution of (2.4) can be written as the sum of $T_t\rho_0$ and the stationary profile $\overline{\rho}(\cdot)$, see also Corollary 3.5.

Definition 2.3. Let $\Delta : \mathscr{S} \to \mathscr{S}$ be the Laplacian operator which is defined on $f \in \mathscr{S}$ as

$$\Delta f(u) = \begin{cases} \partial_u^2 f(u)\,, & \text{if } u \in (0,1)\,, \\ \partial_u^2 f(0^+)\,, & \text{if } u = 0\,, \\ \partial_u^2 f(1^-)\,, & \text{if } u = 1\,. \end{cases} \tag{2.5}$$

Above, $\partial_u^2 f(a^\pm)$ denotes the side limits at the point a. The definition of the operator $\nabla : \mathscr{S} \to C^\infty[0,1]$ is analogous.

We will also use the notations ∂_u and ∂_u^2 for ∇ and Δ, respectively.

Definition 2.4. Let $\mathscr{S}'$ be the topological dual of $\mathscr{S}$ with respect to the topology generated by the seminorms

$$\|f\|_k = \sup_{u\in[0,1]} |\partial_u^k f(u)|\,, \tag{2.6}$$

where $k \in \mathbb{N} \cup \{0\}$. In other words, $\mathscr{S}'$ consists of all linear functionals $f : \mathscr{S} \to \mathbb{R}$ which are continuous with respect to all the seminorms $\|\cdot\|_k$.

In order to avoid topological issues we fix once and for all a finite time horizon T. Let $\mathscr{D}([0,T],\mathscr{S}')$ (resp. $\mathscr{C}([0,T],\mathscr{S}')$) be the space of trajectories which are right continuous, with left limits (resp. continuous) and taking values in $\mathscr{S}'$.

2.3.2. The density fluctuation field

Fix an initial measure μ_n in Ω_n. For $x \in \Sigma_n$ and $t \geq 0$, let

$$\rho_t^n(x) = \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)]\,. \tag{2.7}$$

We extend this definition to the boundary by setting

$$\rho_t^n(0) = \alpha \text{ and } \rho_t^n(n) = \beta\,, \text{ for all } t \geq 0\,. \tag{2.8}$$

A simple computation shows that $\rho_t^n(\cdot)$ is a solution of the discrete equation given by

$$\begin{cases} \partial_t \rho_t^n(x) = \big(n^2 \mathscr{B}_n \rho_t^n\big)(x)\,, & x \in \Sigma_n\,, \quad t \geq 0\,, \\ \rho_t^n(0) = \alpha\,, \quad t \geq 0\,, \\ \rho_t^n(n) = \beta\,, \quad t \geq 0\,, \end{cases} \tag{2.9}$$

where the operator $\mathscr{B}_n$ acts on functions $f : \Sigma_n \cup \{0,n\} \to \mathbb{R}$ as

$$(\mathscr{B}_n f)(x) = \sum_{y=0}^{n} \xi_{x,y}^n \big(f(y) - f(x)\big)\,, \quad \text{for } x \in \Sigma_n\,, \tag{2.10}$$

where

$$\xi^n_{x,y} = \begin{cases} 1\,, & \text{if } |y-x| = 1 \text{ and } x, y \in \Sigma_n\,, \\ \dfrac{1}{n}\,, & \text{if } x = 1, y = 0 \text{ and } x = n-1\,, y = n\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Remark 2.3. If $\mu_n := \mu_{ss}$ is the non-equilibrium stationary measure of the system, then the profile $\rho^n_t(\cdot)$ defined in (2.7) is also stationary in time. In this context, we denote it by $\rho^n_{ss}(\cdot)$. As one can see in [1, Lemma 3.1], $\rho^n_{ss}(\cdot)$ is given by

$$\rho^n_{ss}(x) = a_n\, x + b_n \quad \text{ for } x \in \Sigma_n\,, \tag{2.11}$$

where $a_n = \frac{\beta-\alpha}{3n-2}$ and $b_n = a_n(n-1) + \alpha$. If we extend the definition of $\rho^n_{ss}(\cdot)$ to the boundary of Σ_n, as in (2.8) we get that $\rho^n_{ss}(\cdot)$ is the stationary solution of (2.9).

Now we define the non-equilibrium density fluctuation field as follows.

Definition 2.5 (*Density Fluctuation Field*)**.** We define the density fluctuation field $\mathscr{Y}^n_\cdot$ as the time-trajectory of linear functionals acting on functions $f \in \mathscr{S}$ as

$$\mathscr{Y}^n_t(f) = \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} f(\tfrac{x}{n}) \Big(\eta_{tn^2}(x) - \rho^n_t(x) \Big)\,. \tag{2.12}$$

2.3.3. Non-equilibrium fluctuations

In the next result we assume the following conditions on the initial state μ_n.

Assumption 2.1. For each $n \in \mathbb{N}$, the measure μ_n is associated to a measurable profile $\rho_0 : [0, 1] \to [0, 1]$ in the sense of (2.3).

Assumption 2.2. There exists a constant $C_1 > 0$ not depending on n such that

$$\max_{x\in\Sigma_n} \big|\, \rho^n_0(x) - \rho_0(\tfrac{x}{n}) \big| \le \frac{C_1}{n}\,.$$

Assumption 2.3. There exists a constant $C_2 > 0$ not depending on n such that for

$$\varphi^n_0(x, y) = \mathbb{E}_{\mu_n}[\eta(x)\eta(y)] - \rho^n_0(x)\rho^n_0(y) \tag{2.13}$$

it holds that

$$\max_{1\le x<y\le n-1} \big|\varphi^n_0(x, y)\big| \le \frac{C_2}{n}\,.$$

For each $n \ge 1$, let Q_n be the probability measure on $\mathscr{D}([0, T], \mathscr{S}')$ induced by the density fluctuation field $\mathscr{Y}^n_\cdot$ and the measure μ_n.

Theorem 2.4 (*Non-equilibrium Fluctuations*)**.** *The sequence of measures $\{Q_n\}_{n\in\mathbb{N}}$ is tight on $\mathscr{D}([0, T], \mathscr{S}')$ and all limit points Q are probability measures concentrated on paths $\mathscr{Y}_\cdot$ satisfying*

$$\mathscr{Y}_t(f) = \mathscr{Y}_0(T_t f) + \mathscr{W}_t(f)\,, \tag{2.14}$$

for any $f \in \mathscr{S}$. Above T_t is the semigroup given in Definition 2.2 and $\mathscr{W}_t(f)$ is a mean zero Gaussian variable of variance

$$\int_0^t \|\nabla T_{t-r} f\|^2_{L^2(\rho_r)} dr \,, \tag{2.15}$$

where for $r > 0$

$$\begin{aligned}\langle f, g\rangle_{L^2(\rho_r)} = &\big[\alpha + (1-2\alpha)\rho(r,0)\big]\, f(0)g(0) + \big[\beta + (1-2\beta)\rho(r,1)\big]\, f(1)g(1)\\ &+ \int_0^1 2\chi(\rho(r,u))\, f(u)g(u)\, du \,,\end{aligned} \tag{2.16}$$

$\rho(t,u)$ is the solution of the hydrodynamic equation (2.4), and $\chi(u) = u(1-u)$. Moreover, $\mathscr{Y}_0$ and $\mathscr{W}_t$ are uncorrelated in the sense that $\mathbb{E}_Q\Big[\mathscr{Y}_0(f)\,\mathscr{W}_t(g)\Big] = 0$ for all $f, g \in \mathscr{S}$.

Since $\mathscr{Y}_0(f)$ and $\mathscr{W}_t(g)$ are uncorrelated and have Gaussian distributions, it is natural to ask if they are independent. However, this is not a simple question. One reason for this is the fact that $\mathscr{S}$ is not a Hilbert space (see [12] for Gaussian measures on Hilbert spaces, for instance). We believe that, in fact, $\mathscr{Y}_0(f)$ and $\mathscr{W}_t(g)$ are independent and we conjecture that this would be enough to derive the uniqueness of $\mathscr{Y}_t$. If the latter holds, then the convergence of the sequence of fluctuation fields would follow.

Theorem 2.5 (*Ornstein–Uhlenbeck Limit*)**.** *Assume that the sequence of initial density fields $\{\mathscr{Y}_0^n\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to a mean-zero Gaussian field $\mathscr{Y}$ with covariance given on $f, g \in \mathscr{S}$ by*

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\Big[\mathscr{Y}_0^n(f)\mathscr{Y}_0^n(g)\Big] = \mathbb{E}\Big[\mathscr{Y}_0(f)\mathscr{Y}_0(g)\Big] := \sigma(f,g)\,. \tag{2.17}$$

Then, the sequence $\{Q_n\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to a generalized Ornstein–Uhlenbeck (O.U.) process, which is the formal solution of the equation:

$$\partial_t \mathscr{Y}_t = \Delta \mathscr{Y}_t dt + \sqrt{2\chi(\rho_t)}\nabla W_t \,, \tag{2.18}$$

where W_t is a space–time white noise of unit variance and Δ, ∇ are given in Definition 2.3. As a consequence, the covariance of the limit field $\mathscr{Y}_t$ is given on $f, g \in \mathscr{S}$ by

$$E\,[\mathscr{Y}_t(f)\mathscr{Y}_s(g)] = \sigma(T_t f, T_s g) + \int_0^s \langle \nabla T_{t-r} f, \nabla T_{s-r} g\rangle_{L^2(\rho_r)} dr \,. \tag{2.19}$$

In Section 5.1 we present the precise definition of such generalized O.U. process. As a consequence of the previous result we obtain the non-equilibrium fluctuations starting from a Local Gibbs state.

Corollary 2.6 (*Local Gibbs State*)**.** *Fix a Lipschitz profile $\rho_0 : [0,1] \to [0,1]$ and suppose to start the process from a Bernoulli product measure given by $\mu_n\{\eta : \eta(x) = 1\} = \rho_0(\frac{x}{n})$. Then, Theorem 2.5 remains in force and the covariance in this case is given on $f, g \in \mathscr{S}$ by*

$$E\,[\mathscr{Y}_t(f)\mathscr{Y}_s(g)] = \int_0^1 \chi(\rho_0(u))\, T_t f(u) T_s g(u)\, du + \int_0^s \langle \nabla T_{t-r} f, \nabla T_{s-r} g\rangle_{L^2(\rho_r)} dr \,, \tag{2.20}$$

where $\rho(t,u)$ is the solution of the hydrodynamic equation (2.4) with initial condition given by $\rho_0(\cdot)$.

Remark 2.7. From Theorem 2.5 to prove the last result, it is enough to show the convergence at the initial time, that is:

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\Big[\mathscr{Y}_0^n(f)\mathscr{Y}_0^n(g)\Big] = \int_0^1 \chi(\rho_0(u))\, f(u)g(u)\,du\,,$$

which can be easily verified by means of the convergence of characteristic functions, in the same way of [14, page 297, Cor. 2.2]. We leave the details to the reader.

2.3.4. Stationary fluctuations

Fix $\alpha \neq \beta$. Consider the process starting from the stationary measure μ_{ss}. Note that the density fluctuation field defined on (2.12) is simply given on $f \in \mathscr{S}$ by

$$\mathscr{Y}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} f(\tfrac{x}{n})\Big(\eta_{tn^2}(x) - \rho_{ss}^n(x)\Big)\,, \tag{2.21}$$

where $\rho_{ss}^n(x)$ is defined in (2.7) with $\mu_n = \mu_{ss}$ and given explicitly in (2.11).

Theorem 2.8 (*Stationary Fluctuations*)**.** *Suppose to start the process from μ_{ss} with $\alpha \neq \beta$. Then, $\mathscr{Y}^n$ converges to the centered Gaussian field $\mathscr{Y}$ with covariance given on $f, g \in \mathscr{S}$ by:*

$$\begin{aligned} E_{\mu_{ss}}[\mathscr{Y}(f)\mathscr{Y}(g)] &= \int_0^1 \chi(\overline{\rho}(u)) f(u)g(u)\,du - \Big(\frac{\beta-\alpha}{3}\Big)^2 \int_0^1 [(-\Delta)^{-1} f(u)]g(u)\,du \\ &+ \frac{2(2\beta+\alpha)(2\beta-1)}{3}\int_0^\infty T_t f(1)T_t g(1)\,dt + \frac{2(\beta+2\alpha)(2\alpha-1)}{3}\int_0^\infty T_t f(0)T_t g(0)\,dt\,, \end{aligned} \tag{2.22}$$

with $\overline{\rho}(u) = \big(\frac{\beta-\alpha}{3}\big)\,u + \frac{\beta+2\alpha}{3}$, which is the stationary solution of (2.4).

The time integrals above are well defined in view of the fast decaying of the semigroup T_t, see Corollary 3.3.

We interpret the covariance formula above in the following way: the first term at the right hand side of (2.22) corresponds to the covariance associated to $\overline{\rho}$ in the bulk; the second term corresponds to the covariance associated to $\partial_u \overline{\rho} = (\beta - \alpha)/3$ also in the bulk. The third and fourth terms are associated to $\overline{\rho}$ at the boundaries. Note that for the particular value $\alpha = 1/2$ (or $\beta = 1/2$) the corresponding boundary term vanishes.

3. Semigroup results

In this section we present some useful results about the hydrodynamic equation. We start with the homogeneous version of (2.4), i.e., considering $\alpha = \beta = 0$ as displayed below:

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,, \quad \text{(a)}\\ \partial_u \rho(t,0) = \rho(t,0)\,, & \text{for } t > 0\,, \quad \text{(b)}\\ \partial_u \rho(t,1) = -\rho(t,1)\,, & \text{for } t > 0\,, \quad \text{(c)}\\ \rho(0,u) = \rho_0(u)\,, & u \in [0,1]\,. \quad \text{(d)} \end{cases} \tag{3.1}$$

Proposition 3.1. *Suppose that $\rho_0 \in L^2[0,1]$. Then the previous equation has a solution given by*

$$(T_t\rho_0)(u) := \sum_{n=1}^{\infty} a_n\, e^{-\lambda_n t}\ \Psi_n(u)\,, \tag{3.2}$$

where $\{\Psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2[0,1]$ constituted by eigenfunctions of the associated Regular Sturm–Liouville problem (see (3.3)(a)–(c) below), and a_n are the Fourier coefficients of ρ_0 in that basis. In particular, we prove that $\lambda_n \sim n^2\pi^2$, and as a consequence, the series (3.2) converges exponentially fast, implying that $(T_t\rho_0)(u)$ is smooth in space and time for $t > 0$. Moreover, if we assume $\rho_0 \in \mathscr{S}$, then $(T_t\rho_0)(u)$ will be C^∞ in space and time for $t \geq 0$.

Proof. We start the proof with the associated *Regular Sturm–Liouville Problem*, for details on this subject we refer to [2], for instance. For $\lambda \in \mathbb{R}$, consider the following second-order ordinary differential equation:

$$\begin{cases} \Psi''(u) + \lambda\,\Psi(u) = 0\,, \quad u \in (0,1)\,, & \text{(a)}\\ \Psi(0) = \Psi'(0)\,, & \text{(b)}\\ \Psi(1) = -\Psi'(1)\,. & \text{(c)} \end{cases} \tag{3.3}$$

We claim that there is no solution for $\lambda \leq 0$ aside from the null function. In the case $\lambda = 0$, any solution Ψ of (3.3)(a) must be a line and from the boundary conditions (3.3)(b) and (c), Ψ is the null function. In the case $\lambda < 0$, any solution of (3.3)(a) is of the form $\Psi(u) = C_1 e^{u\sqrt{-\lambda}} + C_2 e^{-u\sqrt{-\lambda}}$, which is a consequence of the fact that the vectorial space of solutions has dimension two. To ease notation, we write $p = \sqrt{-\lambda}$. Applying the boundary conditions (3.3)(b) and (c) we obtain the linear system

$$\begin{cases} (1-p)C_1 + (1+p)C_2 = 0\,, & \text{(a)}\\ (e^p + pe^p)C_1 + (e^{-p} - pe^{-p})C_2 = 0\,, & \text{(b)} \end{cases} \tag{3.4}$$

whose discriminant is given by

$$\Delta(p) = (1-p)^2 e^{-p} - (1+p)^2 e^p < 0\,,$$

for any $p > 0$. In such a case, we get $C_1 = C_2 = 0$, which means that $\Psi(u)$ is the null function.

For $\lambda > 0$, the general solution of (3.3)(a) is of the form $\Psi(u) = A\sin(\sqrt{\lambda}\,u) + B\cos(\sqrt{\lambda}\,u)$. Then, the boundary condition (3.3)(b) implies $B = \sqrt{\lambda}\,A$. To avoid the null solution, we henceforth impose $A \neq 0$. On the other hand, the boundary condition (3.3)(c) gives

$$A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) = -\Big[A\sqrt{\lambda}\cos(\sqrt{\lambda}) - A(\sqrt{\lambda})^2\sin(\sqrt{\lambda})\Big],$$

which becomes the transcendental equation

$$\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda - 1}\,, \tag{3.5}$$

for which there exists a countable number of solutions.

For each $n \in \mathbb{N}$, let λ_n be the solution of (3.5) satisfying $(n-1)\pi \leq \sqrt{\lambda_n} \leq n\pi$, see Fig. 2. Thus $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and $\lambda_n \sim n^2\pi^2$ as $n \to \infty$. Denote now

$$\Psi_n(u) = A_n \sin(\sqrt{\lambda_n}\,u) + A_n\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\,u)\,, \tag{3.6}$$

where A_n is a normalizing constant in such a way Ψ_n has unitary $L^2[0,1]$-norm.

Note that Ψ_n is the solution of the Sturm–Liouville problem above associated to the eigenvalue $-\lambda_n$. Moreover, the set $\{\Psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2[0,1]$. The orthogonality comes from the fact the associated Sturm–Liouville operator is self-adjoint and a proof of it can be found in [2, page 303, Theorem 1]. The proof of completeness can be found in [2, page 363, Section 9].

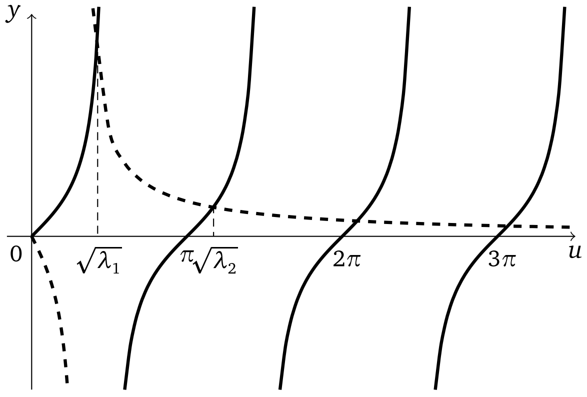

Fig. 2. In solid line, the graph of $f(u) = \tan(u)$. In dashed line, the graph of $g(u) = 2u/(u^2 - 1)$.

Since now we have an orthonormal basis of eigenfunctions, we will apply the heuristics of the classical method of separation of variables to obtain the solution of (3.1)(a)–(d). We start supposing that the solution has the form $\rho(u,t) = \Psi(u)\Phi(t)$. By (3.1)(a) we get that

$$\frac{\Phi'(t)}{\Phi(t)} = -\lambda = \frac{\Psi''(u)}{\Psi(u)},$$

for some $\lambda \in \mathbb{R}$. The first equality in the previous display gives $\Phi(t) = ce^{-\lambda t}$ for some $c \in \mathbb{R}$, while the second equality, under the boundary conditions in (3.1)(c) and (d), corresponds to the Sturm–Liouville problem stated above. Therefore, the solution we seek will be of the form $ce^{-\lambda_n t}\,\Psi_n(u)$. Assuming that $\rho_0 \in L^2[0,1]$, the completeness of the basis permits to write

$$\rho_0(u) = \sum_{n=1}^{\infty} a_n\,\Psi_n(u)\,, \tag{3.7}$$

where $a_n = \langle \rho_0\,,\,\Psi_n\rangle$ are the Fourier coefficients, and $\langle\cdot,\cdot\rangle$ is the inner product w.r.t. $L^2[0,1]$. Now it is a simple task to check that the solution of (3.1)(a)–(d) is given by $\rho(t,u) = \sum_{n=1}^{\infty} a_n\,e^{-\lambda_n t}\;\Psi_n(u)$.

Under the assumption that $\rho_0 \in L^2[0,1]$, by the Cauchy–Schwarz inequality, we have that $|a_n| = |\langle\rho_0,\,\Psi_n\rangle_{L^2[0,1]}| \le \|\rho_0\|_{L^2[0,1]}$. Therefore, the Fourier coefficients are bounded in absolute value. In view of $\lambda_n \sim n^2\pi^2$, this implies that the series in (3.2) converges exponentially fast for $t > 0$, leading to the smoothness in space and time of $(T_t\rho_0)(u)$ for $t > 0$. In order to achieve the smoothness in space and time of $(T_t\rho_0)(u)$ for $t \ge 0$, the previous boundedness on the coefficients a_n is not enough: the assumption $\rho_0 \in \mathscr{S}$ now plays a role. Recalling that the associated Sturm–Liouville operator d^2/du^2 is self-adjoint, we have that

$$a_n = \langle\rho_0,\,\Psi_n\rangle_{L^2[0,1]} = \frac{1}{-\lambda_n}\langle\rho_0,\,-\lambda_n\,\Psi_n\rangle_{L^2[0,1]} = \frac{1}{-\lambda_n}\Big\langle\frac{d^2\rho_0}{du^2},\,\Psi_n\Big\rangle_{L^2[0,1]}. \tag{3.8}$$

Again by the Cauchy–Schwarz inequality, we conclude that $|a_n| \le \frac{1}{\lambda_n}\|\frac{d^2\rho_0}{du^2}\|_{L^2[0,1]} \le \frac{c_1}{n^2}$ for some constant $c_1 > 0$ which does not depend on n. This estimate on a_n assures that the series in (3.7) converges in the supremum norm, which can be used to show that $T_t\rho_0$ defined in (3.2) is C^0 in space and time for $t \ge 0$ (details are omitted here). Repeating the trick of (3.8) and noticing that $\frac{d^2\rho_0}{du^2} \in \mathscr{S}$, we obtain that

$$a_n == \frac{1}{-\lambda_n}\Big\langle \frac{d^2\rho_0}{du^2},\, \Psi_n \Big\rangle_{L^2[0,1]} = \frac{1}{(-\lambda_n)^2}\Big\langle \frac{d^2\rho_0}{du^2},\, -\lambda_n\, \Psi_n \Big\rangle_{L^2[0,1]}$$
$$= \frac{1}{(-\lambda_n)^2}\Big\langle \frac{d^4\rho_0}{du^4},\, \Psi_n \Big\rangle_{L^2[0,1]}.$$

The Cauchy–Schwarz inequality then yields $|a_n| \leq \|\frac{d^4\rho_0}{du^4}\|_{L^2[0,1]}/\lambda_n^2 \leq \frac{c_2}{n^4}$ for some constant $c_2 > 0$ which does not depend on n. This can be used to show that $T_t\rho_0$ is C^1 in space and time for $t \geq 0$. With this idea in mind, an induction procedure can be made, showing that $T_t\rho_0$ is C^∞ in space and time for $t \geq 0$. This completes the proof. □

The previous proposition implies the next results, which play an important role when showing the uniqueness of the associated generalized Ornstein–Uhlenbeck process of Theorem 2.5.

Corollary 3.2. *If $f \in L^2[0,1]$, then for any $t > 0$ we have that $T_t f \in \mathscr{S}$ and $\Delta T_t f \in \mathscr{S}$.*

Proof. Since Ψ_n is a linear combination of sine and cosine, and the conditions of Definition 2.1 are linear, then $\Psi_n \in \mathscr{S}$ for all $n \in \mathbb{N}$. This property is inherited by $T_t f$ and $\Delta T_t f$ due to the explicit formula (3.2) and its exponential convergence. □

Corollary 3.3. *For any $f \in L^2[0,1]$, we have $\lim_{t\to\infty} T_t f = 0$ in the supremum norm. Moreover, this convergence is exponentially fast.*

Proof. This is a consequence of the formula (3.2) and the fact obtained in the proof of Proposition 3.1 that the Fourier coefficients a_n are bounded under the assumption that $f \in L^2[0,1]$. □

Note that, in particular, Corollaries 3.2 and 3.3 hold for any $f \in \mathscr{S}$.

Corollary 3.4. *The operator $\Delta : \mathscr{S} \to \mathscr{S}$ is a bijection. Moreover, for any $f \in \mathscr{S}$,*

$$(a) \quad (-\Delta)^{-1} f(u) = \int_0^{+\infty} T_t f(u)\, dt\,,$$

$$(b) \quad \lim_{t\to\infty} \int_0^t \int_0^1 2\left(T_r f(u)\right)^2 du\, dr = \int_0^1 f(u)(-\Delta)^{-1} f(u)\, du\,.$$

Proof. First of all, notice that all the time integrals above are well defined due to Corollary 3.3. We start by showing (a). Let $f \in \mathscr{S}$ and write $f = \sum_{n=1}^{\infty} a_n\, \Psi_n$. We claim that

$$\Delta \int_0^\infty T_t f(u)\, dt = \Delta \int_0^\infty \Big(\sum_{n=1}^\infty a_n e^{-\lambda_n t}\, \Psi_n(u)\Big)\, dt = \Delta \sum_{n=1}^\infty \Big(\int_0^\infty a_n e^{-\lambda_n t}\, \Psi_n(u)\Big) dt$$
$$= \Delta \sum_{n=1}^\infty \big(\frac{a_n}{\lambda_n}\big)\, \Psi_n(u) = \sum_{n=1}^\infty \big(\frac{a_n}{\lambda_n}\big) \Delta\, \Psi_n(u) = -\sum_{n=1}^\infty a_n\, \Psi_n(u) = -f(u)\,,$$

which will prove (a). Let us make rigorous each step of the sequence of equalities above.

The first equality is due to Proposition 3.1. For the second equality, denote $f_M(t) = \sum_{n=1}^{M} a_n e^{-\lambda_n t} \Psi_n(u)$. To interchange the sum and the integral, we want to apply the Dominated Convergence Theorem, so it is necessary to prove that $|f_M(t)| \leq g(t)$ for some non-negative function g such that $\int_0^\infty g(t)dt < \infty$. Recall from the proof of Proposition 3.1 that, for each $k \in \mathbb{N}$ there exists $c_1 > 0$ such that $|a_n| \leq c_1/n^2$. Thus,

$$|f_M(t)| \leq \sum_{n=1}^{M} |a_n e^{-\lambda_n t} \Psi_n(u)| \leq e^{-\lambda_1 t} \sum_{n=1}^{\infty} \frac{C}{n^2}\,,$$

which gives the desired bound and guarantees the second equality. The third equality is simply integration. In the fourth equality we have used the fast decaying of a_n that we obtained in the proof of Proposition 3.1. In the fifth equality, have used the fact that $-\lambda_n$ is the eigenvalue associated to the eigenfunction Ψ_n. This finishes the proof of (a).

Now we prove (b). Since $\{\Psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2[0,1]$, again by Proposition 3.1 we have that $\langle T_r f, T_r f\rangle = \langle f, T_{2r} f\rangle$. Therefore,

$$\int_0^t \int_0^1 2\left(T_r f(u)\right)^2 du\, dr = \int_0^t \int_0^1 2\, f(u) T_{2r} f(u)\, du\, dr = \int_0^{2t} \int_0^1 f(u) T_r f(u)\, du\, dr.$$

Now, first apply Fubini's Theorem, then take the limit as $t \to \infty$ and recall (a). This finishes the proof. □

Corollary 3.5. *Let $\overline{\rho}(\cdot)$ be the stationary solution of (2.4). Then, the solution $\rho(t,\cdot)$ of (2.4) is given by*

$$\rho(t,u) = \overline{\rho}(u) + T_t\left(\rho_0 - \overline{\rho}\right)(u)\,, \quad u \in [0,1]\,,\ t \geq 0\,,$$

where T_t is the semigroup previously described. In particular, the solution $\rho(t,\cdot)$ of (2.4) is smooth in space and time.

Proof. First we note that from [1, Theorem 2.2] we have, for $u \in [0,1]$, that

$$\overline{\rho}(u) = \left(\frac{\beta-\alpha}{3}\right)u + \left(\alpha + \frac{\beta-\alpha}{3}\right).$$

Then, the time derivative of the function $\overline{\rho}(\cdot)$ is null, as well as its second derivative in space. On other hand, $\overline{\rho}(\cdot)$ satisfies the required non-homogeneous boundary conditions. The result then easily follows by a direct verification. □

Corollary 3.6. *Let $\rho(t,\cdot)$ be the solution of (2.4). Then, for any $u \in [0,1]$,*

$$\lim_{t\to\infty} \rho(t,u) = \overline{\rho}(u)\,,$$

in the supremum norm.

Proof. Immediate from Corollaries 3.5 and 3.3. □

4. Proof of Theorem 2.4

In this section we prove Theorem 2.4 which is the main result of this paper. We start with the martingale decomposition of the process.

4.1. Martingale decomposition

Let $\phi : [0,T]\times[0,1]\to\mathbb{R}$ be a test function. We refer to [14, p. 330] for a proof that

$$M_t^n(\phi) := \mathscr{Y}_t^n(\phi) - \mathscr{Y}_0^n(\phi) - \int_0^t \Lambda^n(\phi)\,ds\,, \tag{4.1}$$

$$N_t^n(\phi) := (M_t^n(\phi))^2 - \int_0^t \Gamma_s^n(\phi)\,ds \tag{4.2}$$

are martingales with respect to the natural filtration $\mathscr{F}_t := \sigma(\eta_s : s\le t)$, where

$$\Lambda^n(\phi) := (\partial_s + n^2\mathscr{L}_n)\mathscr{Y}_s^n(\phi)\,,$$

$$\Gamma_s^n(\phi) := n^2\mathscr{L}_n\mathscr{Y}_s^n(\phi)^2 - 2\mathscr{Y}_s^n(\phi)n^2\mathscr{L}_n\mathscr{Y}_s^n(\phi)\,.$$

By a long but elementary computation,

$$\begin{aligned}\Lambda^n(\phi) =&\mathscr{Y}_s^n(\partial_s\phi) + \frac{1}{\sqrt{n}}\sum_{x=1}^{n-1}\Delta_n\phi(\tfrac{x}{n})\Big(\eta_{sn^2}(x)-\rho_s^n(x)\Big)\,ds\\ &+\sqrt{n}\Big[\nabla_n^+\phi(0)-\phi\big(\frac{1}{n}\big)\Big]\Big(\eta_{sn^2}(1)-\rho_s^n(1)\Big)\\ &+\sqrt{n}\Big[\phi(\tfrac{n-1}{n})+\nabla_n^-\phi(1)\Big]\Big(\eta_{sn^2}(n-1)-\rho_s^n(n-1)\Big)\,.\end{aligned} \tag{4.3}$$

Note that the second term at the right hand side of the previous expression is $\mathscr{Y}_s^n(\Delta_n\phi)$. Above, we have used the notations

$$\Delta_n\phi(x) = n\Big[\phi(\tfrac{x+1}{n})+\phi(\tfrac{x-1}{n})-2\phi(\tfrac{x}{n})\Big],$$

$$\nabla_n^+\phi(x) = n\Big[\phi(\tfrac{x+1}{n})-\phi(\tfrac{x}{n})\Big] \quad\text{and}\quad \nabla_n^-\phi(x) = n\Big[\phi(\tfrac{x}{n})-\phi(\tfrac{x-1}{n})\Big].$$

Also by direct computations we get that

$$\begin{aligned}\Gamma_s^n(\phi) =& \frac{1}{n}\sum_{x=1}^{n-2}\Big(\nabla_n^+\phi\big(\frac{x}{n}\big)\Big)^2\Big(\eta_{sn^2}(x)-\eta_{sn^2}(x+1)\Big)^2\\ &+\Big(\phi\big(\frac{1}{n}\big)\Big)^2\Big(\alpha-2\alpha\eta_{sn^2}(1)+\eta_{sn^2}(1)\Big)+\Big(\phi\big(\frac{n-1}{n}\big)\Big)^2\\ &\Big(\beta-2\beta\eta_{sn^2}(n-1)+\eta_{sn^2}(n-1)\Big)\,.\end{aligned} \tag{4.4}$$

4.2. Proof of Theorem 2.4

Lemma 4.1. *For $\phi\in\mathscr{S}$, the sequence of martingales $\{M_t^n(\phi); t\in[0,T]\}_{n\in\mathbb{N}}$ converges in the topology of $\mathscr{D}([0,T],\mathbb{R})$, as $n\to\infty$, towards a mean-zero Gaussian process $\mathscr{W}_t(\phi)$ with quadratic variation given by*

$$\begin{aligned}\int_0^t\Big\{\int_0^1 2\chi(\rho(r,u))\big(\nabla\phi(u)\big)^2du+\big[\alpha+(1-2\alpha)\rho(r,0)\big]\big(\phi(0)\big)^2\\ +\big[\beta+(1-2\beta)\rho(r,1)\big]\big(\phi(1)\big)^2\Big\}\,dr\,,\end{aligned} \tag{4.5}$$

where $\rho(t,u)$ is the solution of the hydrodynamic equation (2.4).

Proof. To prove this lemma we can apply [11, Theorem VIII.3.12]. We note that by Assertion VIII.3.5 in [11], both [$\hat{\delta}_5$-D] and (3.8) are consequence of

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\Big[\sup_{s\le t}\big|M^n_s(\phi) - M^n_{s-}(\phi)\big|\Big] = 0.$$

The last result can be proved by noting that when a jump occurs, the configuration η only changes its value in at most two sites, hence we have that

$$\sup_{s\le t}\big|M^n_s(\phi) - M^n_{s-}(\phi)\big| = \sup_{s\le t}\big|\mathscr{Y}^n_s(\phi) - \mathscr{Y}^n_{s-}(\phi)\big| \le \tfrac{2\|\phi\|_\infty}{\sqrt{n}},$$

from where the limit follows. Moreover, [γ_5-D] (defined in 3.3 page 470 of [11]) is a consequence of the following argument. In view of (4.4) and recalling Assumption 2.1, we may apply [1, Lemma 5.7] and [1, Theorem 2.7] and standard arguments to conclude that

$$\int_0^t \Gamma^n_s(\phi) ds\,, \tag{4.6}$$

which is an *additive functional* of the exclusion process η_t, converges in distribution, as $n \to \infty$, towards (4.5). Moreover, since the expression above is deterministic, the convergence holds, in fact, in probability. This finishes the proof. □

Remark 4.2. We point out that expression (4.5) simply writes as

$$\int_0^t \|\nabla\phi\|^2_{L^2(\rho_r)}\, dr\,,$$

provided $\phi \in \mathscr{S}$ ($\phi(0) = \nabla\phi(0)$ and $\phi(1) = -\nabla\phi(1)$) for any time $r \in [0, T]$, see (2.16).

As a consequence of the previous result, for each t, the random variable $\mathscr{W}_t(\phi)$ is Gaussian with mean zero and with variance $\int_0^t \|\nabla\phi\|^2_{L^2(\rho_r)} dr$.

Moreover, the random variables $\mathscr{W}_t(f)$ and $\mathscr{Y}_0(g)$ are uncorrelated for any $f, g \in \mathscr{S}$. In fact, $\mathbb{E}\big[\mathscr{W}_t(f)\mathscr{Y}_0(g)\big] = \mathbb{E}\Big[\mathscr{Y}_0(g)\, \mathbb{E}\big[\mathscr{W}_t(f)|\mathscr{F}_0\big]\Big] = 0$, since $\mathscr{W}_0(f) = 0$.

The proof of tightness is postponed to Section 7. From this point on (in this subsection), fix $t \in [0, T]$ and restrict the processes to the time interval $[0, t]$. Choose now the particular test function

$$\phi(u, s) := (T_{t-s} f)(u)\,, \tag{4.7}$$

where T_t is given in Definition 2.2 and $f \in \mathscr{S}$. Note that ϕ is well-defined for all $s \in [0, t]$ and that $\phi \in \mathscr{S}$ in view of Corollary 3.2. For this choice of the test function, (4.3) writes as

$$\begin{aligned}\Lambda^n(T_{t-s} f) =& \mathscr{Y}^n_s\Big(\Delta_n T_{t-s} f - \Delta T_{t-s} f\Big) + \mathscr{Y}^n_s\Big(\Delta T_{t-s} f + \partial_s T_{t-s} f\Big)\\ &+ \sqrt{n}\,\Big(\nabla^+_n(T_{t-s} f)(0) - (T_{t-s} f)\big(\tfrac{1}{n}\big)\Big)\cdot\big(\eta_{sn^2}(1) - \rho^n_s(1)\big)\\ &+ \sqrt{n}\,\Big(\nabla^-_n(T_{t-s} f)(1) - (T_{t-s} f)\big(\tfrac{n-1}{n}\big)\Big)\cdot\big(\eta_{sn^2}(n-1) - \rho^n_s(n-1)\big)\,.\end{aligned} \tag{4.8}$$

We claim that $\Lambda^n(T_{t-s} f)$ goes to zero as $n \to \infty$. Let us examine the four terms at the right hand side of (4.8). By Proposition 3.1, we know that $T_{t-s} f$ is smooth, hence $\Delta_n T_{t-s} f - \Delta T_{t-s} f$ is of order $O(n^{-2})$, which implies that $\mathscr{Y}^n_t(\Delta_n T_{t-s} f - \Delta T_{t-s} f) = O(n^{-3/2})$. The second term at the right hand side of (4.8) is identically zero, since $\partial_s T_{t-s} f = -\Delta T_{t-s} f$. The third and fourth terms at the right hand side of (4.8) go to zero as $n \to \infty$, because $T_{t-s} f$ satisfies the boundary conditions (3.1)(b) and (c), respectively. This proves the claim, which implies also that $\int_0^t \Lambda_n(T_{t-s}f)\,ds$ goes to zero. In other words, by choosing (4.7), the integral term in (4.1) vanishes in the limit as $n\to\infty$.

Let us now look at (4.1) for a fixed time $t\in[0,T]$ and for the choice (4.7). As a consequence of the previous results, together with tightness which is proved in Section 7, any limit point of the sequence $\{\mathscr{Y}_t^n(f)\}_{n\in\mathbb{N}}$ must be of the form

$$\mathscr{Y}_t(f)=\mathscr{Y}_0(T_tf)+\mathscr{W}_t(f)\,, \tag{4.9}$$

where $\mathscr{Y}_0(T_tf)$ and $\mathscr{W}_t(f)$ are uncorrelated and $\mathscr{W}_t(f)$ is a mean zero Gaussian variable of variance given by (2.15). This finishes the proof of Theorem 2.4.

5. Proof of Theorem 2.5

In this section we start by showing the uniqueness of the Ornstein–Uhlenbeck process solution of (2.18) by a martingale problem and then we prove Theorem 2.5.

5.1. Uniqueness of the Ornstein–Uhlenbeck process

Proposition 5.1. *There exists a unique random element $\mathscr{Y}$ taking values in the space $\mathscr{C}([0,T],\mathscr{S}')$ such that:*

(i) *For every function $f\in\mathscr{S}$,*

$$W_t(f):=\mathscr{Y}_t(f)-\mathscr{Y}_0(f)-\int_0^t\mathscr{Y}_s(\Delta f)ds\,, \tag{5.1}$$

$$N_t(f):=\big(W_t(f)\big)^2-\int_0^t\|\nabla f\|^2_{L^2(\rho_r)}\,dr \tag{5.2}$$

are martingales with respect to the filtration $\mathscr{F}_t:=\sigma(\mathscr{Y}_s(g);\,s\le t,\,g\in\mathscr{S})$.

(ii) *$\mathscr{Y}_0$ is a Gaussian field of mean zero and covariance given on $f,g\in\mathscr{S}$ by*

$$\mathbb{E}\big[\mathscr{Y}_0(f)\mathscr{Y}_0(g)\big]=\sigma(f,g)\,, \tag{5.3}$$

where σ was defined in (2.17).

Under the conditions above we have that: for each $f\in\mathscr{S}$, the process $\{\mathscr{Y}_t(f)\,;\,t\ge0\}$ is Gaussian. Moreover, for $s<t$ the distribution of $\mathscr{Y}_t(f)$ conditionally to $\mathscr{F}_s$ is normal of mean $\mathscr{Y}_s(T_{t-s}f)$ and variance $\int_s^t\|\nabla T_{t-r}f\|^2_{L^2(\rho_r)}dr$.

Before proving the proposition we make some comments. The existence of the random element $\mathscr{Y}$ is a consequence of tightness, which is proved in Section 7. The fact that (5.1) and (5.2) are martingales motivates us to call the random element $\mathscr{Y}$ the *formal* solution of (2.18). From this formal equation (2.18) the random element $\mathscr{Y}$ coins the name *generalized Ornstein–Uhlenbeck*. We strongly emphasize that Eq. (2.18) is solely formal and that the norm $\|\cdot\|_{L^2(\rho_r)}$ plays a role in the definition of this generalized Ornstein–Uhlenbeck process.

The next lemma is the key in the proof of Proposition 5.1.

Lemma 5.2. *For any $f\in\mathscr{S}$, $T_{t+\varepsilon}f-T_tf=\varepsilon\,\Delta T_tf+o(\varepsilon,t)$, where $o(\varepsilon,t)$ denotes a function in $\mathscr{S}$ such that $\lim_{\varepsilon\searrow0}\frac{o(\varepsilon,t)}{\varepsilon}=0$ holds in the topology of $\mathscr{S}$. Moreover, the limit is uniform in compact time intervals.*

Proof. The proof is a direct consequence of the explicit formula (3.2). Notice that the inclusion $o(\varepsilon, t) \in \mathscr{S}$ is immediate from Corollary 3.2. Details are omitted here. □

Proof of Proposition 5.1. The structure of proof is the same of [14, page 307]. By (5.1) and (5.2), we have that

$$W_t(f)\cdot\left(\frac{1}{t}\int_0^t \|\nabla f\|^2_{L^2(\rho_r)}dr\right)^{-\frac{1}{2}}$$

is a standard Brownian motion.

Fix $f \in \mathscr{S}$ and $s > 0$. By Itô's Formula (see [19, Theorem. 3.3 and Cor. 3.3]) and the previous comment, the process $\{X_t^s(f)\,;\ t \geq s\}$ defined by

$$X_t^s(f) = \exp\left\{\frac{1}{2}\int_s^t \|\nabla f\|^2_{L^2(\rho_r)}dr + i\left(\mathscr{Y}_t(f) - \mathscr{Y}_s(f) - \int_s^t \mathscr{Y}_r(\Delta f)\,dr\right)\right\}$$

is a (complex) martingale. Fix $S > 0$. We claim now that the process $\{Z_t\,;\ 0 \leq t \leq S\}$ defined by

$$Z_t = \exp\left\{\frac{1}{2}\int_0^t \|\nabla T_{S-r} f\|^2_{L^2(\rho_r)}\,dr + i\,\mathscr{Y}_t(T_{S-t} f)\right\}$$

is also a complex martingale. To prove this claim, consider two times $0 \leq t_1 < t_2 \leq S$ and a partition of the interval $[t_1, t_2]$ in n intervals of equal size, that is, $t_1 = s_0 < s_1 < \cdots < s_n = t_2$, with $s_{j+1} - s_j = (t_2 - t_1)/n$. Observe that

$$\begin{aligned}
&\prod_{j=0}^{n-1} X_{s_{j+1}}^{s_j}(T_{S-s_j} f)\\
&\quad = \exp\left\{\sum_{j=0}^{n-1}\frac{1}{2}\int_{s_j}^{s_{j+1}} \|\nabla T_{S-s_j} f\|^2_{L^2(\rho_r)}dr\right.\\
&\qquad \left. + \, i\sum_{j=0}^{n-1}\Big(\mathscr{Y}_{s_{j+1}}(T_{S-s_j} f) - \mathscr{Y}_{s_j}(T_{S-s_j} f) - \int_{s_j}^{s_{j+1}} \mathscr{Y}_r(\Delta T_{S-s_j} f)\,dr\Big)\right\}.
\end{aligned}$$

As $n \to +\infty$, the first sum inside the exponential above converges to

$$\frac{1}{2}\int_{t_1}^{t_2} \|\nabla T_{S-r} f\|^2_{L^2(\rho_r)}\,dr\,,$$

due to the smoothness of the semigroup T_t. The second sum inside the exponential can be rewritten as

$$\begin{aligned}
&\mathscr{Y}_{t_2}(T_{S-t_2+\frac{(t_2-t_1)}{n}} f) - \mathscr{Y}_{t_1}(T_{S-t_1} f)\\
&\quad + \sum_{j=1}^{n-1}\Big(\mathscr{Y}_{s_j}(T_{S-s_{j-1}} f - T_{S-s_j} f) - \int_{s_j}^{s_{j+1}}\mathscr{Y}_r(\Delta T_{S-s_j} f)\,dr\Big).
\end{aligned} \tag{5.4}$$

Note that, by Lemma 5.2 the previous sum can be written as

$$\sum_{j=1}^{n-1}\Big(\int_{s_j}^{s_{j+1}}\Big(\mathscr{Y}_{s_j}(\Delta T_{S-s_j} f) - \mathscr{Y}_r(\Delta T_{S-s_j} f)\Big)\,dr + \mathscr{Y}_{s_j}\big(o(\tfrac{t_2-t_1}{n}, S - s_j)\big)\Big).$$

Keep in mind that $f \in \mathscr{S}$ and $t \in [0,T]$. By Proposition 3.1, the function $t \mapsto T_t f$ is C^∞, which in particular says that the function $t \mapsto \Delta T_t f$ is uniformly continuous. This, together with $\mathscr{Y} \in \mathscr{C}([0,T],\mathscr{S}')$ imply that $(s,t) \mapsto \mathscr{Y}_s(\Delta T_t f)$ is continuous for fixed $f \in \mathscr{S}$. The time horizon considered is the compact set $[0,T]$, hence the function $(s,t) \mapsto \mathscr{Y}_s(\Delta T_t f)$ is uniformly continuous for fixed $f \in \mathscr{S}$.

The last uniform continuity shows that the integrand function above vanishes and from this we obtain that (5.4) converges almost surely to $\mathscr{Y}_{t_2}(T_{S-t_2}f) - \mathscr{Y}_{t_1}(T_{S-t_1}f)$. Hence we have proved that

$$\lim_{n\to+\infty}\prod_{j=0}^{n-1} X^{s_j}_{s_{j+1}}(T_{S-s_j}f)$$
$$= \exp\left\{\frac{1}{2}\int_{t_1}^{t_2}\|\nabla T_{S-r}f\|^2_{L^2(\rho_r)}dr + i\Big(\mathscr{Y}_{t_2}(T_{S-t_2}f) - \mathscr{Y}_{t_1}(T_{S-t_1}f)\Big)\right\},$$

which is equal to $\frac{Z_{t_2}}{Z_{t_1}}$ almost surely. Since the complex exponential is bounded, the Dominated Convergence Theorem gives additionally the L^1 convergence, which implies

$$\mathbb{E}\Big[G\,\frac{Z_{t_2}}{Z_{t_1}}\Big] = \lim_{n\to+\infty}\mathbb{E}\Big[G\prod_{j=0}^{n-1} X^{s_j}_{s_{j+1}}(T_{S-s_j}f)\Big],$$

for any measurable bounded function G. Take G bounded and $\mathscr{F}_{t_1}$-measurable. Since for any $f \in \mathscr{S}$ the process $X^s_t(f)$ is a martingale, we take the conditional expectation with respect to $\mathscr{F}_{s_{n-1}}$, and we are led to

$$\mathbb{E}\Big[G\prod_{j=0}^{n-1} X^{s_j}_{s_{j+1}}(T_{S-s_j}f)\Big] = \mathbb{E}\Big[G\prod_{j=0}^{n-2} X^{s_j}_{s_{j+1}}(T_{S-s_j}f)\Big].$$

By induction, we conclude that

$$\mathbb{E}\Big[G\,\frac{Z_{t_2}}{Z_{t_1}}\Big] = \mathbb{E}\Big[G\Big],$$

for any G bounded and $\mathscr{F}_{t_1}$-measurable, proving that $\{Z_t\,;\, t \geq 0\}$ is, in fact, a martingale. From $\mathbb{E}[Z_t|\mathscr{F}_s] = Z_s$, we get

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^t \|\nabla T_{S-r}f\|^2_{L^2(\rho_r)}dr + i\,\mathscr{Y}_t(T_{S-t}f)\Big\}\Big|\mathscr{F}_s\Big]$$
$$= \exp\Big\{\frac{1}{2}\int_0^s \|\nabla T_{S-r}f\|^2_{L^2(\rho_r)}dr + i\,\mathscr{Y}_s(T_{S-s}f)\Big\},$$

which in turn gives

$$\mathbb{E}\Big[\exp\Big\{i\,\mathscr{Y}_t(T_{S-t}f)\Big\}\Big|\mathscr{F}_s\Big] = \exp\Big\{-\frac{1}{2}\int_s^t \|\nabla T_{S-r}f\|^2_{L^2(\rho_r)}dr + i\,\mathscr{Y}_s(T_{S-s}f)\Big\}.$$

Note that $T_{S-s}f = T_{t-s}T_{S-t}f$. Thus, writing $g = T_{S-t}f$ we get

$$\mathbb{E}\Big[\exp\Big\{i\,\mathscr{Y}_t(g)\Big\}\Big|\mathscr{F}_s\Big] = \exp\Big\{-\frac{1}{2}\int_s^t \|\nabla T_{t-r}g\|^2_{L^2(\rho_r)}dr + i\,\mathscr{Y}_s(T_{t-s}g)\Big\}.$$

Replacing back g by λf, where $\lambda \in \mathbb{R}$, we obtain

$$\mathbb{E}\Big[\exp\Big\{i\,\lambda\,\mathscr{Y}_t(f)\Big\}\Big|\mathscr{F}_s\Big] = \exp\Big\{-\frac{\lambda^2}{2}\int_s^t \|\nabla T_{t-r} f\|^2_{L^2(\rho_r)} dr + i\,\lambda\,\mathscr{Y}_s(T_{t-s} f)\Big\},$$

which means that, conditionally to $\mathscr{F}_s$, the random variable $\mathscr{Y}_t(f)$ has Gaussian distribution of mean $\mathscr{Y}_s(T_{t-s} f)$ and variance $\int_s^t \|\nabla T_{t-r} f\|^2_{L^2(\rho_r)} dr$. Since the distribution at time zero is determined by (5.3), by successively conditioning we get the uniqueness of the finite dimensional distributions of the process $\{\mathscr{Y}_t(f)\,;\ t \in [0, T]\}$, which assures uniqueness in law of the random element $\mathscr{Y}$. □

5.2. *Characterization of limit points*

We prove here that any limit point of $\{Q_n\}_{n\in\mathbb{N}}$ is concentrated on solutions of (2.18), i.e., the limit satisfies (i) and (ii) of Proposition 5.1. Fix a test function $f \in \mathscr{S}$ (note that f does not depend on time) and let us look at the martingale (4.1) taking $\phi = f$. Lemma 4.1 guarantees the convergence of the martingale $M^n_t(f)$ towards a mean-zero Gaussian process $\mathscr{W}_t(f)$, whose quadratic variation is given by $\int_0^t \|\nabla f\|^2_{L^2(\rho_r)} dr$, see Remark 4.2. By the hypothesis of Theorem 2.5, $\{\mathscr{Y}^n_0\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to a mean-zero Gaussian field $\mathscr{Y}_0$ with covariance given by (2.17). Thus $\{\mathscr{Y}^n_0(f)\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to $\mathscr{Y}_0(f)$ as well. By tightness proved in Section 7, we can pick a subsequence of $\mathbb{N}$ such that $\{\mathscr{Y}^n_t;\, t \in [0, T]\}_{n\in\mathbb{N}}$ is convergent in the Skorohod topology of $\mathscr{D}([0, T], \mathscr{S}')$ as $n \to \infty$. Therefore, $\{\mathscr{Y}^n_t(f);\, t \in [0, T]\}_{n\in\mathbb{N}}$ also converges in the Skorohod topology of $\mathscr{D}([0, T], \mathbb{R})$, as $n \to \infty$. By abuse of notation, we denote this subsequence by n. Let us look to the integral part of the martingale $M^n_t(f)$. Due to (4.3),

$$\int_0^t n^2 \mathscr{L}_n \mathscr{Y}^n_s(f)\, ds = \int_0^t \Big\{\mathscr{Y}^n_s(\Delta f) + \mathscr{R}^n_s(f)\Big\}\, ds\,,$$

where

$$\begin{aligned}\mathscr{R}^n_s(f) = \mathscr{Y}^n_s\Big(\Delta_n f - \Delta f\Big) &+ \sqrt{n}\,\Big(\nabla^+_n f(0) - f\big(\tfrac{1}{n}\big)\Big)\cdot\big(\eta_{sn^2}(1) - \rho^n_s(1)\big)\\ &+ \sqrt{n}\,\Big(\nabla^-_n f(1) + f\big(\tfrac{n-1}{n}\big)\Big)\cdot\big(\eta_{sn^2}(n-1) - \rho^n_s(n-1)\big)\,.\end{aligned}$$

Since $f \in \mathscr{S}$, it follows that

$$\lim_{n\to\infty} \mathscr{R}^n_s(f) = 0\,.$$

On the other hand, by Corollary 3.2 we know that $\Delta f \in \mathscr{S}$, which together with the convergence of $\mathscr{Y}^n_t$ gives us that

$$\lim_{n\to\infty} \int_0^t \mathscr{Y}^n_s(\Delta f)\, ds = \int_0^t \mathscr{Y}_s(\Delta f)\, ds\,,$$

so that

$$W_t(f) = \mathscr{Y}_t(f) - \mathscr{Y}_0(f) - \int_0^t \mathscr{Y}_s(\Delta f)\, ds\,,$$

concluding the characterization of limit points.

Proof of Theorem 2.5. The convergence follows from Proposition 5.1, the previous characterization of limit points and tightness proved in Section 7. It remains only to prove that the covariance is as given in (2.19). By (2.14) we have that (2.19) is a consequence of the fact that $\mathscr{W}_t$ is Gaussian of variance (2.15) and that $\mathscr{Y}_0$ and $\mathscr{W}_t$ are uncorrelated. This finishes the proof. □

Remark 5.3. We note that above we took test functions in the space $\mathscr{S}$. These functions have boundary conditions of Robin type, namely the same boundary conditions that appear at the level of the hydrodynamics but taking the parameters $\alpha = \beta = 0$. If we do not take $\mathscr{S}$ as the space of test functions then the boundary terms in $\mathscr{R}_t^n$ might not vanish and then we would have to control their variance. We believe that the variance of these terms is of order one and for that reason in (5.1) there would be extra terms corresponding to those boundary terms. In that case we would need to derive another characterization of the Ornstein–Uhlenbeck process.

6. Proof of Theorem 2.8

Before presenting the proof we give some description about the argument. First we check that the conditions that we impose on the initial measure in Theorem 2.4 are fulfilled by the stationary measure μ_{ss}. Then, from Theorem 2.4 we conclude that the sequence $\{\mathscr{Y}_n\}_{n\in\mathbb{N}}$ is tight and that all limits points satisfy (2.14) where $\mathscr{W}_t(f)$ is a mean zero Gaussian variable of variance

$$\begin{aligned}&\int_0^t\int_0^1 2\chi(\overline{\rho}(u))(\nabla T_s f(u))^2\,du\,ds\\&+\int_0^t\big[\alpha-(1-2\alpha)\overline{\rho}(0)\big](\nabla T_s f(0))^2+\big[\beta-(1-2\beta)\overline{\rho}(1)\big](\nabla T_s f(1))^2\,ds.\end{aligned}\tag{6.1}$$

Then, we take $\mathscr{Y}_t$ a solution of (2.18), whose covariance is given by (2.19), we compute the asymptotic behavior of this covariance and we prove that it converges to (2.22). This, together with the fact that Corollary 3.5 implies that $\lim_{t\to\infty}T_t f = 0$ in the $L^2[0,1]$ norm, we conclude that the variance of $\mathscr{W}_t(f)$ converges to (2.22). As a consequence of these arguments the proof ends.

Before checking that the stationary measures satisfy the assumptions of Theorem 2.4, we make an observation about the sequence $\rho_{ss}^n(x)$. Consider the stationary solution of (2.4) denoted by $\overline{\rho}:[0,1]\to[0,1]$. By [1, Theorem 2.2] $\overline{\rho}(\cdot)$ is given by $\overline{\rho}(u) = a\,u\, + \,b$, for all $u\in[0,1]$, where $a = \frac{\beta-\alpha}{3}$, and $b = \alpha + \frac{\beta-\alpha}{3}$. By [1, Lemma 3.1], we have $\big|\rho_{ss}^n(x) - \overline{\rho}(\frac{x}{n})\big| \le \frac{C}{n}$, for all $x\in\Sigma_n$, where C does not depend on x. As we extended $\rho_{ss}^n(\cdot)$ to 0 and n as $\rho_{ss}^n(0) = \alpha$ and $\rho_{ss}^n(n) = \beta$, this convergence is not true at $x = 0$ and $x = n$. But it is not a problem in this paper, here it is enough to have the convergence in (0, 1). Moreover, if we needed the convergence in the whole interval [0, 1], we just had to consider the extension of $\rho_{ss}^n(\cdot)$ to 0 and n as $\rho_{ss}^n(0) = b_n$ and $\rho^n(n)_{ss} = a_n n + b_n$. From the previous considerations, Assumption 2.2 is trivially satisfied when μ_n coincides with the stationary measure μ_{ss}. Moreover, from [1, Lemma 3.2] Assumption 2.3 is also valid in this case. From the previous observations we conclude that the result of Theorem 2.4 is true when we start the system from the stationary measure. Now, from Theorem 2.4 we know that the sequence $\{\mathscr{Y}_n\}_{n\in\mathbb{N}}$ is tight, all limits points satisfy (2.14) and $\mathscr{W}_t(f)$ is a mean zero Gaussian variable of variance (6.1). Now we take $\mathscr{Y}_t$ a solution of (2.18), whose covariance is given by (2.19) and we compute the asymptotic behavior of this covariance. We claim that it converges to (2.22). Note that by Corollary 3.5 $\lim_{t\to\infty}T_t f = 0$ in the $L^2[0,1]$ norm. To prove the claim, we note that by the polarization identity it is enough to analyze the variance. For this purpose, fix $f\in\mathscr{S}$ and take $g = f$ and $s = t$ in (2.19) to have that:

$$E[(\mathscr{Y}_t(f))^2] = \sigma(T_t f, T_t f) + \int_0^t \langle\nabla T_{t-r} f, \nabla T_{t-r} f\rangle_{L^2(\rho_r)} dr\ ,$$

where T_t is given in Definition 2.2. By Corollary 3.3, $T_t f$ vanishes as $t \to +\infty$, hence the first term at the right hand side of the previous expression converges to zero, as $t \to \infty$.

Now we analyze the remaining term that we denote by $R_t(f)$ which, by (2.16), is given by

$$\int_0^t \int_0^1 2\chi(\rho(r,u))(\nabla T_{t-r} f(u))^2 \, du \, dr \tag{6.2}$$

$$+ \int_0^t \big(\alpha - (1-2\alpha)\rho(r,0)\big)(\nabla T_{t-r} f(0))^2 \, dr \tag{6.3}$$

$$+ \int_0^t \big(\beta - (1-2\beta)\rho(r,1)\big)(\nabla T_{t-r} f(1))^2 \, dr. \tag{6.4}$$

We start by dealing with the first term above. Performing an integration by parts in space we can rewrite (6.2) as:

$$\int_0^t 2\chi(\rho(r,u))\nabla T_{t-r} f(u) T_{t-r} f(u)\Big|_{u=0}^{u=1} dr$$
$$- \int_0^t \int_0^1 2\nabla\Big(\chi(\rho(r,u))\nabla T_{t-r} f(u)\Big)\, T_{t-r} f(u) \, du \, dr\,.$$

Since T_t is the semigroup associated to the Laplacian operator of Definition 2.2, then $\partial_u T_{t-r} f(0) = T_{t-r} f(0)$ and $\partial_u T_{t-r} f(1) = -T_{t-r} f(1)$. As a consequence, the first term in last expression is equal to

$$- \int_0^t \Big(2\chi(\rho(r,1))(T_{t-r} f(1))^2 + 2\chi(\rho(r,0))(T_{t-r} f(0))^2\Big) \, dr\,, \tag{6.5}$$

while the second term is equal to

$$- \int_0^t \int_0^1 2\nabla\chi(\rho(r,u)) \, \nabla T_{t-r} f(u) \; T_{t-r} f(u) \, du \, dr$$
$$- \int_0^t \int_0^1 2\chi(\rho(r,u)) \; \Delta T_{t-r} f(u) \; T_{t-r} f(u) \, du \, dr.$$

Since $2f\partial_u f = \partial_u f^2$, last expression becomes

$$- \int_0^t\!\int_0^1 \nabla\chi(\rho(r,u))\nabla(T_{t-r} f(u))^2 du\, dr - \int_0^t\!\int_0^1 2\chi(\rho(r,u))(\Delta T_{t-r} f(u))T_{t-r} f(u) du \; dr.$$

On the other hand since $\partial_r T_{t-r} f = -\Delta T_{t-r} f$, we can rewrite the last expression as:

$$- \int_0^t\!\int_0^1 \nabla\chi(\rho(r,u))\nabla(T_{t-r} f(u))^2 du\, dr + \int_0^t\!\int_0^1 2\chi(\rho(r,u))\partial_r T_{t-r} f(u) T_{t-r} h(u) du\, dr,$$

which equals to

$$- \int_0^t \int_0^1 \nabla\chi(\rho(r,u))\nabla(T_{t-r} f(u))^2 \, du \, dr + \int_0^t \int_0^1 \chi(\rho(r,u))\partial_r (T_{t-r} f(u))^2 \, du \, dr\,.$$

Integrating by parts in time the second term above, we write the last expression as

$$
\begin{aligned}
&-\int_0^t\int_0^1 \nabla\chi(\rho(r,u))\,\nabla(T_{t-r}f(u))^2\,du\,dr\\
&+\int_0^1 \chi(\rho(t,u))(f(u))^2du-\int_0^1 \chi(\rho(0,u))(T_tf(u))^2du\\
&-\int_0^t\int_0^1 \partial_r\chi(\rho(r,u))(T_{t-r}f(u))^2\,du\,dr\,.
\end{aligned}
$$

Integrating by parts in space the first term above, then the previous expression is equal to

$$
\begin{aligned}
&-\int_0^t\Big[(1-2\rho(r,1))\nabla\rho(r,1)(T_{t-r}f(1))^2-(1-2\rho(r,0))\nabla\rho(r,0)(T_{t-r}f(0))^2\Big]\,dr\\
&+\int_0^t\int_0^1 \Delta\chi(\rho(r,u))(T_{t-r}f(u))^2\,du\,dr\\
&+\int_0^1 \chi(\rho(t,u))(f(u))^2du-\int_0^1 \chi(\rho(0,u))(T_tf(u))^2du\\
&-\int_0^t\int_0^1 \partial_r\chi(\rho(r,u))(T_{t-r}f(u))^2\,du\,dr\,.
\end{aligned}
$$

Since $-\partial_r\chi(\rho(r,u))+\Delta\chi(\rho(r,u))=-2(\nabla\rho(r,u))^2$, last expression is equal to

$$
\begin{aligned}
&-\int_0^t\Big[(1-2\rho(r,1))\nabla\rho(r,1)(T_{t-r}f(1))^2-(1-2\rho(r,0))\nabla\rho(r,0)(T_{t-r}f(0))^2\Big]\,dr\\
&-\int_0^t\int_0^1 2(\nabla\rho(r,u))^2(T_{t-r}f(u))^2\,du\,dr\\
&+\int_0^1 \chi(\rho(r,u))(f(u))^2du-\int_0^1 \chi(\rho(0,u))(T_tf(u))^2du\,.
\end{aligned}
$$

In short, we have that $R_t(f)$ is the sum of the expression above, (6.5), (6.3) and (6.4). Then, since $\rho(r,u)$ is the solution of the hydrodynamic equation and using the fact that $\partial_uT_{t-r}f(0)=T_{t-r}f(0)$ and $\partial_uT_{t-r}f(1)=-T_{t-r}f(1)$ we can rewrite $R_t(f)$ as

$$
-\int_0^t\int_0^1 2(\nabla\rho(r,u))^2(T_{t-r}f(u))^2\,du\,dr \tag{6.6}
$$

$$
+\int_0^1 \chi(\rho(t,u))(f(u))^2du-\int_0^1 \chi(\rho(0,u))(T_tf(u))^2\,du \tag{6.7}
$$

$$
+\int_0^t\Big[2\rho(r,1)(2\beta-1)(T_{t-r}f(1))^2+2\rho(r,0)(2\alpha-1)(T_{t-r}f(0))^2\Big]\,dr\,. \tag{6.8}
$$

At this point we take the limit of $R_t(f)$ as $t\to+\infty$. By Corollaries 3.3 and 3.6, (6.7) converges to

$$
\int_0^1 \chi(\overline{\rho}(u))(f(u))^2du\,. \tag{6.9}
$$

Denote $g(r,u) = (\nabla\rho(r,u))^2$ for short. By a change of variables in time, (6.6) can be rewritten as

$$-\int_0^t\int_0^1 g(t-r,u)\cdot 2\left(T_r f(u)\right)^2 du\,dr\,. \tag{6.10}$$

From (b) in Corollary 3.4, $\lim_{t\to+\infty} g(t-r,u) = \left(\nabla\overline{\rho}(u)\right)^2$ and $\lim_{r\to+\infty} T_r f = 0$. Then, some analysis permits to conclude that (6.10) converges to

$$-\int_0^1 \left(\nabla\overline{\rho}(u)\right)^2 f(u)\,(-\Delta)^{-1} f(u)\,du\,. \tag{6.11}$$

It remains to deal with (6.8). Since $\lim_{t\to+\infty}\rho(r,1) = \overline{\rho}(1)$, $\lim_{t\to+\infty}\rho(r,0) = \overline{\rho}(0)$ and $\lim_{t\to+\infty} T_t f = 0$, similarly to what we have done above, we deduce that (6.8) converges to

$$\int_0^\infty 2\overline{\rho}(1)(2\beta-1)(T_r f(1))^2\,dr + \int_0^\infty 2\overline{\rho}(0)(2\alpha-1)(T_r f(0))^2\,dr\,. \tag{6.12}$$

In conclusion, the limit of $R_t(f)$ as $t\to+\infty$ is the sum of (6.9), (6.11) and (6.12), that is,

$$\begin{aligned}&-\int_0^1 (\nabla\overline{\rho}(u))^2 f(u)(-\Delta)^{-1} f(u)\,du + \int_0^1 \chi(\overline{\rho}(u))(f(u))^2 du\\ &+ 2\overline{\rho}(1)(2\beta-1)\int_0^\infty (T_t f(1))^2\,dr + 2\overline{\rho}(0)(2\alpha-1)\int_0^\infty (T_t f(0))^2\,dr\,.\end{aligned} \tag{6.13}$$

Since $\nabla\overline{\rho}(u) = \frac{\beta-\alpha}{3}$, $\overline{\rho}(0) = \frac{\beta+2\alpha}{3}$ and $\overline{\rho}(1) = \frac{2\beta+\alpha}{3}$, we have just proved the claim. In particular, the variance of $\mathscr{W}_t(f)$ converges, as $t\to\infty$, to (2.22), so that $\mathscr{W}_t(f)$ converges in distribution to a mean zero Gaussian random variable with variance given by (2.22). Collecting the previous results we get that the random variables $\mathscr{Y}_t(f)$ are mean zero Gaussian with covariance given by (2.22). Since the process is stationary this ends the proof of Theorem 2.8.

7. Tightness

Now we prove that the sequence of processes $\{\mathscr{Y}_t^n; t\in[0,T]\}_{n\in\mathbb{N}}$ is tight. Recall that we have defined the density fluctuation field on test functions $f\in\mathscr{S}$. Since we want to use Mitoma's criterion [16] for tightness, we need the following property from the space $\mathscr{S}$.

Proposition 7.1. *The space $\mathscr{S}$ endowed with the semi-norms given in (2.6) is a Fréchet space.*

Proof. The definition of a Fréchet space can be found, for instance, in [18]. Since $C^\infty([0,1])$ endowed with the semi-norms (2.6) is a Fréchet space, and a closed subspace of a Fréchet space is also a Fréchet space, it is enough to show that $\mathscr{S}$ is a closed subspace of $C^\infty([0,1])$, which is a consequence of the fact that uniform convergence implies point-wise convergence. □

As a consequence of Mitoma's criterion [16] and Proposition 7.1, the proof of tightness of the $\mathscr{S}'$ valued processes $\{\mathscr{Y}_t^n; t\in[0,T]\}_{n\in\mathbb{N}}$ follows from tightness of the sequence of real-valued processes $\{\mathscr{Y}_t^n(f); t\in[0,T]\}_{n\in\mathbb{N}}$, for $f\in\mathscr{S}$.

Proposition 7.2 (*Mitoma's Criterion, [16]*). *A sequence of processes $\{x_t; t\in[0,T]\}_{n\in\mathbb{N}}$ in $\mathscr{D}([0,T],\mathscr{S}')$ is tight with respect to the Skorohod topology if, and only if, the sequence $\{x_t(f); t\in[0,T]\}_{n\in\mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathscr{D}([0,T],\mathbb{R})$, for any $f\in\mathscr{S}$.*

Now, to show tightness of the real-valued process we use the Aldous' criterion:

Proposition 7.3. *A sequence* $\{x_t; t \in [0,T]\}_{n\in\mathbb{N}}$ *of real-valued processes is tight with respect to the Skorohod topology of* $\mathscr{D}([0,T],\mathbb{R})$ *if:*

(i) $\lim_{A\to+\infty} \limsup_{n\to+\infty} \mathbb{P}_{\mu_n}\Big(\sup_{0\le t\le T}|x_t| > A\Big) = 0\,,$
(ii) *for any* $\varepsilon > 0\,,\ \lim_{\delta\to 0} \limsup_{n\to+\infty} \sup_{\lambda\le\delta} \sup_{\tau\in\mathscr{T}_T} \mathbb{P}_{\mu_n}(|x_{\tau+\lambda} - x_\tau| > \varepsilon) = 0\,,$

where $\mathscr{T}_T$ *is the set of stopping times bounded by* T.

Fix $f \in \mathscr{S}$. By (4.1), it is enough to prove tightness of $\{\mathscr{Y}_0^n(f)\}_{n\in\mathbb{N}}$, $\{\int_0^t \Gamma_s^n(f)\,ds; t \in [0,T]\}_{n\in\mathbb{N}}$, and $\{\mathscr{M}_t^n(f); t \in [0,T]\}_{n\in\mathbb{N}}$.

7.1. Tightness at the initial time

To prove that the sequence $\{\mathscr{Y}_0^n(f)\}_{n\in\mathbb{N}}$ is tight, it is enough to observe that

$$\mathbb{E}_{\mu_n}\Big[\Big(\mathscr{Y}_0^n(f)\Big)^2\Big] = \frac{1}{n}\sum_{x=1}^{n-1} f^2\Big(\frac{x}{n}\Big)\chi(\rho_0^n(x)) + \frac{2}{n}\sum_{x<y} f\Big(\frac{x}{n}\Big) f\Big(\frac{y}{n}\Big)\varphi_0^n(x,y)$$

and by Assumption 2.3 last expression is bounded.

7.2. Tightness of the martingales

By Lemma 4.1 since the sequence of martingales converges, in particular, it is tight.

7.3. Tightness of the integral terms

Let us check the first claim of Aldous' criterion for the integral term $\int_0^t \Gamma_s^n(f)\,ds$. Since $f \in \mathscr{S}$ and by the Cauchy–Schwarz inequality we have that

$$\mathbb{E}_{\mu_n}\Big[\sup_{t\le T}\Big(\int_0^t \Gamma_s^n(f)\,ds\Big)^2\Big] \le T\int_0^T \mathbb{E}_{\mu_n}\Big[\Big(\frac{1}{\sqrt{n}}\sum_{x=1}^{n-1}\Delta_n f(\tfrac{x}{n})(\eta_{sn^2}(x) - \rho_s^n(x))\Big)^2\Big]\,ds$$

plus a term $O(n^{-1})$. The term on the right hand side of last expression is bounded from above by T^2 times

$$\frac{1}{n}\sum_{x=1}^{n-1}\Big(\Delta_n f(\tfrac{x}{n})\Big)^2 \sup_{t\le T}\chi(\rho_t^n(x)) + \frac{1}{n}\sum_{\substack{x\ne y\\ x,y=1}}^{n-1} \Delta_n f(\tfrac{x}{n})\Delta_n f(\tfrac{y}{n}) \sup_{t\le T}\varphi_t^n(x,y)\,, \tag{7.1}$$

where $\varphi_t^n(x,y)$ is given in (8.1). Then, by Proposition 8.1 and since $f \in \mathscr{S}$, last expression is bounded by a constant. Now we need to check the second claim. For that purpose, fix a stopping time $\tau \in \mathscr{T}_T$. By the Chebychev's inequality together with (7.1), we get that

$$\mathbb{P}_{\mu_n}\Big(\Big|\int_\tau^{\tau+\lambda}\Gamma_s^n(f)\,ds\Big| > \varepsilon\Big) \le \frac{1}{\varepsilon^2}\mathbb{E}_{\mu_n}\Big[\Big(\int_\tau^{\tau+\lambda}\Gamma_s^n(f)\,ds\Big)^2\Big] \le \frac{\delta^2 C}{\varepsilon^2}\,,$$

which vanishes as $\delta \to 0$.

8. Discrete equations

In this section we prove some technical estimates that are needed along the paper.

8.1. Two-point correlation function

Definition 8.1 (*Two-Point Correlation Function*)**.** For each $x, y \in \Sigma_n$, $x < y$, and $t \in [0, T]$, we define the two-point correlation function as

$$\varphi_t^n(x, y) = \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)\eta_{tn^2}(y)] - \rho_t^n(x)\rho_t^n(y)\,, \tag{8.1}$$

where ρ_t^n was defined in (2.7). Moreover, for $x = 0$ or $y = n$, we set $\varphi_t^n(x, y) = 0$,

Proposition 8.1. *There exists $C > 0$ such that*

$$\sup_{t\geq 0}\max_{(x,y)\in V_n} |\varphi_t^n(x, y)| \leq \frac{C}{n}\,, \tag{8.2}$$

where $V_n = \{(x, y)\,;\ x, y \in \mathbb{N},\ 0 < x < y < n\}$.

Proof. First, observe that $\varphi_t^n(x, y)$ can be rewritten as

$$\mathbb{E}_{\mu_n}[(\eta_{tn^2}(x) - \rho_t^n(x))(\eta_{tn^2}(y) - \rho_t^n(y))]\,,$$

so that from Kolmogorov's forward equation, we have that

$$\partial_t\varphi_t^n(x, y) = \mathbb{E}_{\mu_n}\Big[(n^2\mathscr{L}_n + \partial_t)(\eta_{tn^2}(x) - \rho_t^n(x))(\eta_{tn^2}(y) - \rho_t^n(y))\Big]\,.$$

Applying (2.1) and (2.2) and performing some long, but elementary, calculations we deduce that φ_t^n solves the following system of ODE's:

$$\begin{cases} \partial_t\varphi_t^n(x, y) = n^2\mathscr{A}_n\varphi_t^n(x, y) + g_t^n(x, y)\,, & \text{for } (x, y) \in V_n\,,\ t > 0\,,\\ \varphi_t^n(x, y) = 0\,, & \text{for } (x, y) \in \partial V_n\,,\ t > 0\,,\\ \varphi_0^n(x, y) = \mathbb{E}_{\mu_n}[\eta_0(x)\eta_0(y)] - \rho_0^n(x)\rho_0^n(y)\,, & \text{for } (x, y) \in V_n \cup \partial V_n\,, \end{cases} \tag{8.3}$$

where $\mathscr{A}_n$ is the linear operator that acts on functions $f : V_n \cup \partial V_n \to \mathbb{R}$ as

$$(\mathscr{A}_n f)(u) = \sum_{v\in V_n} c_n(u, v)\big[f(v) - f(u)\big]\,, \qquad \text{for } u \in V_n\,,$$

with

$$c_n(u, v) = \begin{cases} 1\,, & \text{if } \|u - v\| = 1 \text{ and } u, v \in V_n\,,\\ n^{-1}\,, & \text{if } \|u - v\| = 1 \text{ and } u \in V_n,\ v \in \partial V_n\,,\\ 0\,, & \text{otherwise,} \end{cases}$$

and ∂V_n represents the boundary of the set V_n, which we define as

$$\partial V_n = \{(0, 1), \ldots, (0, n)\} \cup \{(1, n), \ldots, (n - 1, n)\}\,,$$

see Fig. 3 for an illustration. Above we have that

$$g_t^n(x, y) = -\,n^2(\rho_t^n(x) - \rho_t^n(x + 1))^2 \cdot \mathbf{1}_{\{\mathscr{D}_n\}}(x, y)\,, \tag{8.4}$$

where the diagonal $\mathscr{D}_n$ is defined by

$$\mathscr{D}_n = \{(x, y) \in V_n;\ y = x + 1\}$$

and $\mathbf{1}_{\{\Gamma\}}$ is the indicator function of the set Γ.

Above, $\|\cdot\|$ denotes the supremum norm. Note that $\mathscr{A}_n$ is the generator of a random walk in $V_n \cup \partial V_n$, , denoted by $\{X_{tn^2};\ t \geq 0\}$, which has jump rates given by $c_n(u, v)$ and is absorbed

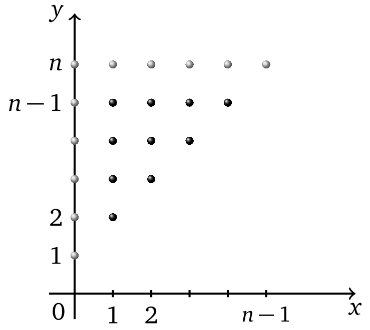

Fig. 3. Black balls are elements of V_n and gray balls are elements of ∂V_n.

at the boundary ∂V_n. We observe that the random walk $\{X_{tn^2};\ t \geq 0\}$ is the two-dimensional analogue of the one-dimensional random walk with generator $\mathscr{B}_n$ given in (2.10). Denote by $\mathbf{P}_u$ and $\mathbf{E}_u$ the corresponding probability and expectation, respectively, starting from the position $u \in V_n$. Now we introduce the function

$$\phi_t^n(x, y) = \mathbf{E}_{(x,y)}\Big[\varphi_0^n(X_{tn^2}) + \int_0^t g_{t-s}^n(X_{sn^2})\,ds\Big], \tag{8.5}$$

with φ_0^n and g_t^n given above. Since $\mathbf{E}_{(x,y)}[f(X_{tn^2})] = (e^{tn^2\mathscr{A}_n} f)(x, y)$ is a semigroup and by using Kolmogorov's forward equation and Leibniz Integral Rule, we can show that the function ϕ_t^n is solution of the semi-linear system (8.3), so that $\phi_t^n = \varphi_t^n$. Therefore, in order to prove the proposition we just have to estimate the two terms at the right hand side of last display, since

$$\max_{(x,y)\in V_n} |\varphi_t^n(x, y)| \leq \max_{(x,y)\in V_n} |\varphi_0^n(x, y)| + \max_{(x,y)\in V_n} \Big|\mathbf{E}_{(x,y)}\Big[\int_0^t g_{t-s}^n(X_{sn^2})\,ds\Big]\Big| .$$

From Assumption 2.3, the first term on the right hand side of last expression is bounded from above by c/n. It remains to deal with the second term. Note that since the operator $n^2\mathscr{A}_n$ is a bounded operator (for n fixed) it generates a uniformly continuous semigroup $\{e^{sn^2\mathscr{A}_n};\ s \geq 0\}$ on $V_n \cup \partial V_n$. By Fubini's Theorem

$$\mathbf{E}_{(x,y)}\Big[\int_0^t g_{t-s}^n(X_{sn^2})\,ds\Big] = \int_0^t \big(e^{sn^2\mathscr{A}_n} g_{t-s}^n\big)(x, y)\,ds\,.$$

Changing variables, the right hand side of last expression can be written as

$$\int_0^t \big(e^{(t-r)n^2\mathscr{A}_n} g_r^n\big)(x, y)\,dr\,.$$

Thus, the proof ends as a consequence of the next lemma. $\square$

Before stating the next lemma, we notice that for $u, v \in V_n \cup \partial V_n$

$$e^{tn^2\mathscr{A}_n}(u, v) = \mathbf{P}_u\Big[X_{tn^2} = v\Big]. \tag{8.6}$$

Lemma 8.2. *There exists $C > 0$ which does not depend on n such that*

$$\sup_{t\geq 0} \max_{(x,y)\in V_n} \Big|\int_0^t \big(e^{(t-r)n^2\mathscr{A}_n} g_r^n\big)(x, y)\,dr\Big| \leq \frac{C}{n}\,.$$

Proof. Since the function g_r^n defined in (8.4) is supported on the diagonal $\mathscr{D}_n$, we can rewrite $(e^{(t-r)n^2\mathscr{A}_n} g_r^n)(x,y)$ as

$$\sum_{z=1}^{n-2} e^{(t-r)n^2\mathscr{A}_n}\big((x,y),\,(z,z+1)\big)\, g_r^n(z,z+1)\,.$$

Then, for all $(x,y)\in V_n$,

$$\Big|\int_0^t \big(e^{(t-r)n^2\mathscr{A}_n} g_r^n\big)(x,y)\,dr\Big| \le S_n\cdot\int_0^t \sum_{z=1}^{n-2} e^{(t-r)n^2\mathscr{A}_n}\big((x,y),\,(z,z+1)\big)\,dr\,, \tag{8.7}$$

where

$$S_n = \sup_{r\ge 0}\max_{z\in\{1,\dots,n-2\}} |g_r^n(z,z+1)|\,. \tag{8.8}$$

First we will estimate the time integral at the right hand side of (8.7) and then we will estimate S_n. By (8.6) together with a change of variables and by the definition of $\mathscr{D}_n$, we get

$$\int_0^{tn^2}\sum_{z=1}^{n-2} \mathbf{P}_{(x,y)}\big[X_s=(z,z+1)\big]\,\frac{ds}{n^2} = \int_0^{tn^2} \mathbf{P}_{(x,y)}\big[X_s\in\mathscr{D}_n\big]\,\frac{ds}{n^2}\,.$$

Extending the interval of integration to infinity and applying Fubini's Theorem on the last integral, we bound it from above by

$$\frac{1}{n^2}\,\mathbf{E}_{(x,y)}\Big[\int_0^\infty \mathbf{1}_{\{X_s\in\mathscr{D}_n\}}\,ds\Big]\,.$$

Note that the expectation above is the total time spent by the random walk $\{X_s;\, s\ge 0\}$ on the diagonal $\mathscr{D}_n$. We claim that

$$\mathbf{E}_{(x,y)}\Big[\int_0^\infty \mathbf{1}_{\{X_s\in\mathscr{D}_n\}}\,ds\Big] \le C\,n\,,$$

for all $(x,y)\in V_n$. To prove the claim, we note that in Section 3 of [1], the authors introduced a coupling to compare a random walk with slow rates at the boundary with the random walk presented in Section 4 of [15]. In equation (4.2) of [15] there is an explicit expression for the total time spent by the random walk on the diagonal $\mathscr{D}_n$. Moreover, if in [1] we take $\theta=1$ the random walk considered there becomes the random walk $\{X_{tn^2}\,;\ t\ge 0\}$ defined above. Then, by the aforementioned coupling we prove the claim.

As a consequence of the previous estimate, the integral at the right hand side of (8.7) is bounded from above by C/n. In order to conclude the proof, we need to prove that S_n, which was defined in (8.8), is bounded. By the definition of g_r^n given in (8.4), it is enough to prove that

$$\big|\rho_t^n(x+1)-\rho_t^n(x)\big| \le \frac{C}{n}\,,$$

for all $x\in\{1,\dots,n-2\}$ and uniformly in $t\ge 0$ and this follows from Proposition 8.3. □

8.2. Estimates for the discrete equation

Proposition 8.3. *Let $\rho_t^n(\cdot)$ be the solution of (2.9). Then, there exists $C>0$ which does not depend on n such that*

$$\big|\rho_t^n(x+1)-\rho_t^n(x)\big| \le \frac{C}{n}\,, \tag{8.9}$$

for all $x\in\{1,\dots,n-2\}$, uniformly in $t\ge 0$.

Proof. Let $\rho(t,u)$ be the solution of the equation

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \partial_u \rho(t,0) = \rho(t,0^+) - \rho(t,0)\,, & \text{for } t>0\,,\\ \partial_u \rho(t,1) = \rho(t,1) - \rho(t,1^-)\,, & \text{for } t>0\,,\\ \rho(t,0)=\alpha\,,\quad \rho(t,1)=\beta\,, & \text{for } t>0\,,\\ \rho(0,u) = \rho_0(u)\,, & u\in[0,1]\,. \end{cases} \tag{8.10}$$

We notice that ρ_t is essentially the solution of the hydrodynamic equation given in (2.4), but discontinuous at 0 and 1. By Corollary 3.5, we have assured the smoothness of $\rho(t,u)$ in $(0,1)$.

Let $\gamma_t^n(x) := \rho_t^n(x) - \rho_t(\frac{x}{n})$ for $x\in\Sigma_n\cup\{0,n\}$. Then γ_t^n satisfies the equation

$$\begin{cases} \partial_t \gamma_t^n(x) = (n^2\mathscr{B}_n\gamma_t^n)(x) + F_t^n(x)\,, & x\in\Sigma_n\,,\quad t\geq 0\,,\\ \gamma_t^n(0)=0\,,\quad \gamma_t^n(n)=0\,,\quad t\geq 0\,, \end{cases} \tag{8.11}$$

where, for $x\in\{2,\dots,n-2\}$, F_t^n accounts for the difference between discrete and continuous Laplacians and, for $x\in\{1,n-1\}$, $F_t^n(x) = (n^2\mathscr{B}_n - \partial_u^2)\rho_t(\frac{x}{n})$.

In order to prove (8.9), we add and subtract $\rho_t\big(\frac{x+1}{n}\big)$ and $\rho_t\big(\frac{x}{n}\big)$ to $|\rho_t^n(x+1)-\rho_t^n(x)|$ and use the triangle inequality to have that

$$\big|\rho_t^n(x+1)-\rho_t^n(x)\big| \leq \big|\gamma_t^n(x+1)\big| + \big|\gamma_t^n(x)\big| + \Big|\rho_t(\tfrac{x+1}{n}) - \rho_t(\tfrac{x}{n})\Big|\,.$$

Since ρ_t is smooth in $(0,1)$, it remains to show that γ_t^n is bounded by c/n. For that purpose, let $\{\mathbb{X}_s,\ s\geq 0\}$ be the random walk on $\Sigma_n\cup\{0,n\}$, with generator $\mathscr{B}_n$, absorbed at the boundaries $\{0,n\}$. Denote by E_x the expectation with respect to the probability induced by the generator $\mathscr{B}_n$ and the initial position x. As before, we can write the solution of (8.11) as

$$\gamma_t^n(x) = E_x\Big[\gamma_0^n(\mathbb{X}_{tn^2}) + \int_0^t F_{t-s}^n(\mathbb{X}_{sn^2})\,ds\Big]\,.$$

Then,

$$\sup_{t\geq 0}\max_{x\in\Sigma_n}|\gamma_t^n(z)| \leq \max_{x\in\Sigma_n}|\gamma_0^n(x)| + \sup_{t\geq 0}\max_{x\in\Sigma_n}\Big|E_x\Big[\int_0^t F_{t-s}^n(\mathbb{X}_{sn^2})\,ds\Big]\Big|\,.$$

Since $\gamma_0^n(x) = |\rho_0^n(x)-\rho_0(x)|$, by Assumption 2.2 we only need to control the second term at the right hand side of the previous expression. Repeating the same strategy as before, we decompose the expectation above into the possible positions of the chain at time s and we are left to estimate

$$\int_0^t \sum_{z=1}^{n-1} P_x\Big[\mathbb{X}_{sn^2}=z\Big]\cdot F_{t-s}^n(z)\,ds\,. \tag{8.12}$$

Since the discrete Laplacian approximates the continuous Laplacian, we conclude that $F_t^n(x)\leq C/n^2$ for any $x\in\{2,\dots,n-2\}$ and for any $t\geq 0$. Therefore, we can bound (8.12) by

$$\frac{C}{n^2} + \sum_{k\in\{1,n-1\}} E_x\Big[\int_0^\infty \mathbf{1}_{\{\mathbb{X}_{sn^2}=k\}}\,ds\Big]\cdot|F_t^n(k)|\,. \tag{8.13}$$

Moreover, we also have that

$$\begin{aligned} F_t^n(1) &= n^2\Big(\rho_t\big(\frac{2}{n}\big) - \rho_t\big(\frac{1}{n}\big)\Big) - n\Big(\rho_t\big(\frac{0}{n}\big) - \rho_t\big(\frac{1}{n}\big)\Big) - \partial_u\rho_t\big(\frac{1}{n}\big)\\ &= n\Big(\partial_u\rho_t\big(\frac{1}{n}\big) - \rho_t\big(\frac{0}{n}\big) - \rho_t\big(\frac{1}{n}\big)\Big) + O(1)\,, \end{aligned}$$

and by the boundary conditions in (8.10) we obtain that $|F_t^n(1)| \leq C$ for any $t \geq 0$. For $k = n-1$ we obtain exactly the same bound as for $k = 1$.

The expectation in (8.13) is the average time spent by the random walk at the site k until its absorption. As an application of the Markov Property, it can be expressed as the solution of the elliptic equation

$$\begin{cases} -\mathscr{B}_n \psi^n(x) = C\delta_{x=k}\,, & \forall\, x \in \Sigma_n\,, \\ \psi^n(0) = 0\,, \quad \psi^n(n) = 0\,, \end{cases}$$

where C is a constant. A simple computation shows that, for $k = 1$,

$$\psi^n(x) = -\frac{1}{3n^2 - 2n}x + \frac{2n-1}{3n^2 - 2n}\,, \qquad \forall\, x \in \Sigma_n\,,$$

so that $\max_{x=1,\ldots,n-1}|\psi^n(x)| \leq C/n$. For $k = n-1$ the same bound holds. Putting all the estimates together, the proof ends. □

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