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Hydrodynamics of TASEP via microscopic characteristics

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Salvador-Bahia
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Dissertação de Mestrado apresentada ao Colegiado da Pós-Graduação em Matemática da Universidade Federal da Bahia como requisito parcial para obtenção do título de Mestre em Matemática.

Orientador: Prof. Dr. Tertuliano Franco Santos Franco.

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Banca examinadora

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À meu pai, minha avó e meus irmãos.

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"Science, my boy, is made up of mistakes, but they are mistakes which it is useful to make, because they lead little by little to the truth."

– Julio Verne

Resumo

Neste trabalho, estudamos o limite hidrodinâmico para o processo de exclusão simples totalmente assimétrico (TASEP) o qual converge para a solução da equação de Burgers; esta convergência pode se ver por meio de dois fatos e para prová-los precisaremos de noções como a partícula marcada, partículas de segunda classe e o fluxo e também alguns resultados sobre estas como a lei dos grandes números e acoplamento. A prova do resultado principal se dividirá em duas partes: o caso *shock* e o caso *rarefação*.

Palavras-chave: TASEP, equação de Burgers, limite hidrodinâmico, acoplamento, partícula marcada e partícula de segunda classe.

Abstract

In this work, we study the hydrodynamic limit for the totally asymmetric simple exclusion process (TASEP) which converges for the solution of Burgers equation; this convergence can be seen by means of two facts and to prove them we will need notions such as the tagged particle, second class particles and the flux and also some results on these as the law of large numbers and coupling. The proof of this main result will be divided into two parts: the *shock* case and the *rarefaction* case.

Keywords: TASEP, Burgers equation, hydrodynamic limit, coupling, tagged particle and second class particle.

Contents

Introduction

The Totally Asymmetric Simple Exclusion Process (TASEP) is a prototypical stochastic model of transport in one dimension. Initially was introduced the ASEP around 50 years ago in parallel in biology [9], [8] and probability theory [13], it has been extensively studied by a variety of methods. The ASEP and TASEP are good models of many physical systems, such as traffic on highways [5], transport in narrow channels [2] and motion of motor proteins on microtubules [4], among others.

The TASEP is defined on a one-dimensional lattice (in our case $\mathbb{Z}$), whose sites may be occupied or not by a particle. During time evolution particles are allowed to hop from the site they occupy towards the site directly to the right, say from site i to $i+1$, provided this latter site is empty (not occupied), each particle waits a random exponent mean one amount of time. We make an graphical construction for this process, which will help us to understand some definitions and proofs.

The inviscid Burgers equation is a first order quasilinear hyperbolic equation, which we will study their solution for some initial conditions and sometimes we obtain cases for the solution such as shock and rarefaction. In particular, we work with the Riemann problem which is the Burgers equation with a particular initial condition, namely, the density has value λ for non positive positions and density ρ for positive positions.

We are establishing that rescaling time and space in the same way for the TASEP, the density of the particles converges to a deterministic function which satisfies the Burgers equation, limit known as the hydrodynamic limit. The hydrodynamic limit meets two important results these are convergence of the density fields and local equilibrium which says that, respectively, which are the limit for the number of particles dividing by t over any interval and the local measure for a given process.

This was first noticed by Rost [11], who dealt with an initial configuration which has only particles at the negative sites and the Burgers equation has initial condition 1 to the left to the origin and 0 in other case. Ferrari [7] has a proof for a large family of distributions, this is the purpose of this dissertation, we will present some of the main ideas and results considered in [7].

The graphical construction of the TASEP induces the definition of other elements, such as the coupling and first and second class particle, these help to prove the hydrodynamic limit via law of large number for some of these elements and previous lemmas.

The Chapter 1 introduces the graphical construction of the TASEP and shows the its invariant measure. Also the Burgers equation, the motivation, some examples with their respective solutions and characterization of some solutions via some criteria. At the end we have the the Burke's theorem for the $(M, M, 1)$ process. It was used for the proof the law of large number for the tagged particle, a very important fact.

The Chapter 2 starts with a heuristic derivation for the hydrodynamic limit. Subsequent, we have the theorem including the hydrodynamic limit. On other hand, defined the tagged particle, first and second class particle, flux and tagged and isolated second class particle. Also, the graphical construction of the coupling and the law of large number for some of the above elements. Finally, using all these elements and results is prove the hydrodynamic limit for the increasing shock and rarefaction fan cases.

Chapter 1

TASEP and Burgers Equation

We start by introducing some required notions to our subject.

1.1 The Totally Asymmetric Simple Exclusion Process

We construct now the *Totally Asymmetric Simple Exclusion Process* (TASEP), which is a standard particle system in Probability and in Statistical Mechanics. The TASEP is a Markov process taking values on $\{0,1\}^{\mathbb{Z}}$, where particles jump only to a fixed direction. Here, we fix once and for all that particles move only to the right at rate 1. Moreover, a particle cannot jump to an occupied site. We will denote this Markov process by $\{\eta_t : t \geq 0\}$. In general, we define the TASEP with respect to a given initial configuration, which may be random.

1.1.1 Construction of the TASEP

We will denoted by η an element on the space $\{0,1\}^{\mathbb{Z}}$, which represents a configuration of particles. By *sites* we mean elements of $\mathbb{Z}$. We denote by ω a realization of independent Poisson processes of parameter one indexed on $\mathbb{Z}$, where each element $(x,t) \in \omega$ we indicate as a arrow from x to $x+1$ at time t, see Figure 1.1.

For a fixed time $T > 0$, by the Borel-Cantelli Lemma, we have that for almost all ω there are infinitely many $x_i \in \mathbb{Z}$ such that $(x_i, t) \notin \omega$ for all $t \in (0,T)$. We can then take a partition $\{x_i\}_{i\in\mathbb{N}}$ of $\mathbb{Z}$ such that do not exist arrows connecting the finite boxes $[x_i+1, x_{i+1}] \cap \mathbb{Z}$ on the time interval $[0,T]$. Since the boxes are of finite length, we can label the arrows inside each box by order of appearance in time, i.e, $(x_i,t) < (x_j,s) \leftrightarrow t < s$ and if $t = s$, $(x_i,t) < (x_j,s) \leftrightarrow x_i < x_j$.

The Markov process can be constructed as follows. Consider ω that satisfies the above property and η an initial configuration of particles, where $\eta(x) = 0$ means the site x is empty and $\eta(x) = 1$ means the site x is occupied. The state η_t will be a function of ω and the initial state η obeying the following rule: for a given box with the property above, we take the first arrow (x,t) if we have in the time t^- that there is a particle at x and no particle at $x+1$, then the particle follows the arrow $x \to x+1$ so in the time t there a is

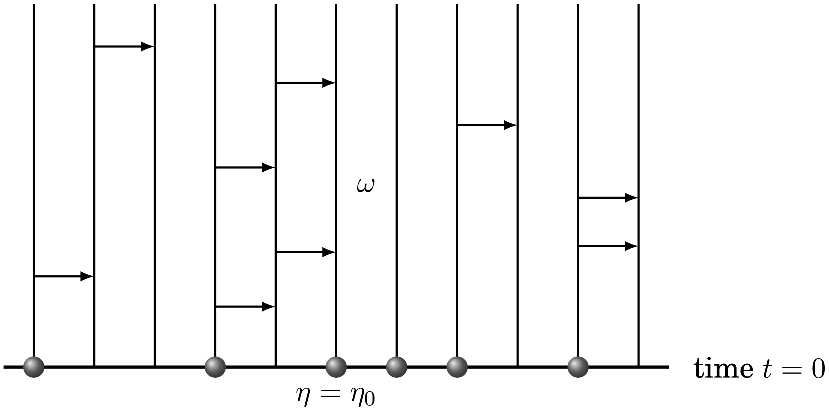

Figure 1.1. ω is the Poisson process represented by the arrows, η is the configuration initial (dots) of the particles.

particle at $x+1$ and no particle at x. If this arrow does not satisfy the previous condition, then we ignore this arrow. We can continue with this process for all the arrows in the box. We repeat this process for each box in order to obtain a particle configuration (see Figure 1.2 for an illustration) depending on the initial stateη and the Poisson realization ω, which we denote by $\eta_t[\eta, \omega]$, where $0 \leq t \leq T$. For greater times that T, let us say between T and $2T$, we consider the initial configuration η_T and the same arrows of ω with times in $[T, 2T]$ and repeat the above process successively. As any time interval $[kT, (k+1)T|]$. In general, we have constructed the process

$$(\eta_t[\eta, \omega] : t \geq 0) .$$

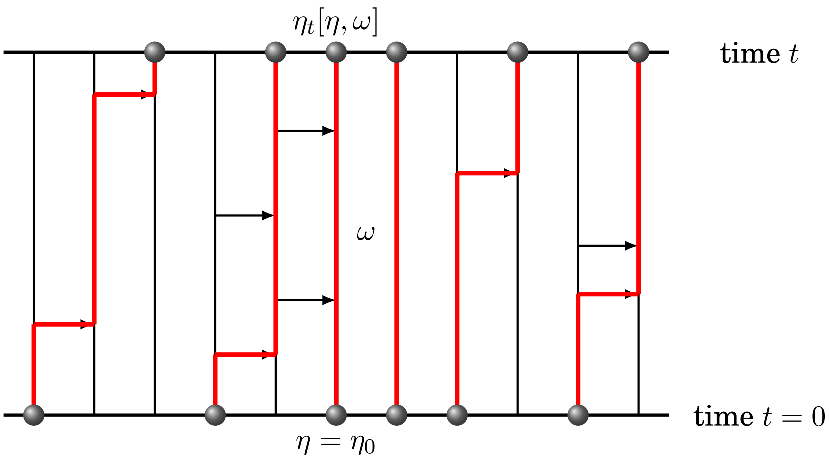

Figure 1.2. ω is a realization of the Poisson processes, η and η_t are the initial configuration and the concigeration at the time t, respectively.

This process satisfies almost surely the property $\eta_{t+s}[\eta, \omega] = \eta_s[\eta_t[\eta, \omega], \tau_t \omega]$, where

$\tau_t\omega := \{(x,s) : (x,t+s) \in \omega\}$ has a same distribution as ω and it is independent of $\omega \cap (\mathbb{Z} \times [0,t])$, by the properties of the Poisson process ω. This implies that the process η_t is Markovian.

1.1.2 Random initial configuration

In this case we want to construct, in a same probability space, different TASEP process starting from different initial configurations. Let $U = \{U(x) : x \in \mathbb{Z}\}$ be a collection of iid random variables uniformly distributed in $[0,1]$. Assume that U is independent of ω. For each $\rho \in [0,1]$, we define $\eta^\rho = \eta^\rho[U]$ given by

$$\eta^\rho(x) := \mathbb{1}_{\{U(x)<\rho\}}. \tag{1.1}$$

Note that $\eta^\rho(x)$ has distribution Ber(ρ) for each $x \in \mathbb{Z}$, so the distribution of η^ρ is Bernoulli product. Under the definition of η^ρ we can define η^ρ_t as follow

$$\eta^\rho_t := \eta_t[\eta^\rho, \omega],$$

as the configuration at the time t considering the initial configuration η^ρ. Note that η^ρ_t is a function of U and ω. Finally, we define

$$f_A(\eta) := \prod_{x\in A} \eta(x).$$

If ν is a random configuration in $\{0,1\}^{\mathbb{Z}}$, then $\{\mathbb{E}[f_A(\nu)] : A \subseteq \mathbb{Z},\ A \text{ finite}\}$ characterizes the distribution of ν. In particular, the distribution of η^ρ is characterized by $\mathbb{E}\big[f_A(\eta^\rho)\big] = \rho^{|A|}$, where $|A|$ is the cardinal of A.

Lemma 1.1. *For each $\rho \in [0,1]$, the distribution of η^ρ is invariant for the TASEP. That is, for any $A \subset \mathbb{Z}$ finite we have that $\mathbb{E}[f_A(\eta^\rho_t)] = \rho^{|A|}$, for all $t \geq 0$.*

Proof. This result is standard and can be found in [12, Example 6.2, page 79] for instance. We sketch below an alternative proof avoiding to speak about infinitesimal generators.

Fix $\rho \in [0,1]$ and consider the TASEP type process $\{\eta^N(t) : t \geq 0\}$ on $\Omega_N = \{0,1\}^{-N,-N+1,\dots,N}$ with the following dynamics at the boundary: at rate ρ a particle tries to enter the system at the site $-N$ (doing it successfully only, and only if, that site is empty at that moment). Moreover, at rate $(1-\rho)$, if there is a particle at the site N at that moment, this particle leaves the system.

This process $\{\eta^N(t) : t \geq 0\}$ is a finite state one and it possible to check that the Bernoulli product measure μ of constant parameter ρ is invariant for it by checking that

$$\sum_{\eta'} \mu(\eta')\lambda(\eta',\eta) = \mu(\eta)\sum_{\eta'} \lambda(\eta',\eta), \tag{1.2}$$

where $\lambda(\eta',\eta)$ are the above mentioned rates (including the movement to the right of particles in the bulk, which is everywhere equal to one). In fact, the reader can verify that checking (1.2) for the aforementioned process corresponds to check that

$$\rho^k(1-\rho)^{2N-k}\#\{\text{ consecutive pairs } 1,0 \text{ in } \eta\ \}$$
$$+\ \rho\mathbb{1}_{\{\eta(-N)=0\}}\rho^{k-1}(1-\rho)^{2N-k+1} + (1-\rho)\mathbb{1}_{\{\eta(N)=1\}}\rho^{k+1}(1-\rho)^{2N-k-1}$$
$$=\ \rho^k(1-\rho)^{2N-k}\Big\{\#\{\text{ consecutive pairs } 1,0 \text{ in } \eta\ \} + \rho\mathbb{1}_{\{\eta(-N)=0\}} + (1-\rho)\mathbb{1}_{\{\eta(N)=1\}}\Big\},$$

which is straightforward.

At this point, a coupling argument as in the proof of Proposition 2.1 ahead and the limit as $N \to \infty$ allows us to conclude the measure μ is invariant for the TASEP on $\mathbb{Z}$. □

1.2 The Burgers equation

The equation was first introduced by Harry Bateman [1] in 1915 derived in a physical context and later studied in 1948 by Johannes Martinus Burgers [3]. The Burgers equation is a fundamental nonlinear partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, gas dynamics, and traffic flow.

1.2.1 Motivation

Consider a street starting at point x_1 and ending at point x_2. Let $u(x,t)$ be the density of cars at point x and time t. Then, the number of cars between the points x_1 and x_2 at time t is given by

$$\int_{x_1}^{x_2} u(x,t)dx.$$

Now, the rate of change in the numbers of cars between x_1 and x_2 at time t is represented by

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = f(u(x_1,t)) - f(u(x_2,t)),$$

where f represents the flow rate off the street. Assuming that u and f are continuously differentiable functions, we have that

$$\int_{x_1}^{x_2} u_t(x,t)dx = f(u(x_1,t)) - f(u(x_2,t)),$$

and, therefore,

$$\frac{1}{x_2-x_1}\int_{x_1}^{x_2} u_t(x,t)dx = \frac{f(u(x_1,t)) - f(u(x_2,t))}{x_2-x_1}.$$

Taking the limit as $x_2 \to x_1$, we get

$$u_t = -[f(u)]_x.$$

Therefore, we can say that the density of cars at point x at time t satisfies the following partial differential equation (PDE):

$$u_t + [f(u)]_x = 0\,, \text{ for some smooth function } f. \tag{1.3}$$

However, this was done assuming the density of the cars was a continuous function. We would like to derive some sort of notion to say that a function u which is not even differentiable will solve the PDE.

1.2.2 Solution of Burgers equation

Let us consider the PDE, called the one-dimensional forced Burgers equation, given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial f(u)}{\partial x} + \nu \dfrac{\partial^2 u}{\partial x^2} + F(x,t), & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) := u_0(x) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $\nu \geq 0$ is the diffusion coefficient, $F(x,t)$ is a forcing of system and $\phi(x)$ a initial condition of the PDE. In our case we study the *inviscid Burgers equation*, i.e, we consider the particular case in (1.4) where $\nu = 0$, $F(x,t) \equiv 0$ and f a smooth function, which is a usual case seen in [10]. Hence, the corresponding system is

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial f(u)}{\partial x}, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) := u_0(x) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

Note that this PDE is the same one mentioned in (1.3) with the initial condition ϕ. We say that u is *strong* or *classical solution* of (1.5) if u is continuously differentiable and u satisfies (1.5). Sometimes this solution does not exist. Thus, we want to allow solutions which are not differentiable or not even continuous. In this way we introduce now the notion of weak solution of a PDE.

We say that a function $v : \mathbb{R}^n \to \mathbb{R}$ has *compact support* if $v \equiv 0$ outside of some compact set of $\mathbb{R}^n$. A function u is said a *weak solution* of (1.5) if

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x]\, dx\, dt + \int_{-\infty}^\infty \phi(x)v(x,0)dx = 0, \tag{1.6}$$

for all smooth functions $v \in C^\infty(\mathbb{R} \times [0,+\infty))$ of compact support.

Theorem 1.1. *If u is a strong solution of (1.5), then u is a weak solution of (1.5).*

Proof. If u is a classical solution of (1.5), then u is continuously differentiable and satisfies

$$\begin{cases} u_t + [f(u)]_x = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) := \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

Moreover, for any function $v : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ with compact support,

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty [u_t + [f(u)]_x]\, v dx\, dt \\ &= \int_0^\infty \int_{-\infty}^\infty u_t v dx\, dt + \int_0^\infty \int_{-\infty}^\infty [f(u)]_x v dx\, dt. \end{aligned} \tag{1.8}$$

Integrating by parts each integral above and using the fact that v vanishes at infinity, we have

$$\int_{-\infty}^\infty [f(u)]_x v dx = vf(u)\Big|_{-\infty}^{\infty} - \int_{-\infty}^\infty f(u)v_x dx = -\int_{-\infty}^\infty f(u)v_x dx,$$

therefore

$$\int_0^\infty \int_{-\infty}^\infty [f(u)]_x v dx\, dt = -\int_0^\infty \int_{-\infty}^\infty f(u)v_x dx\, dt,$$

and

$$\int_0^\infty \int_{-\infty}^\infty u_t v dx\, dt = \int_0^\infty \int_{-\infty}^\infty uvdx\Big|_0^\infty - \int_0^\infty \int_{-\infty}^\infty uv_t dx\, dt$$
$$= -\int_{-\infty}^\infty \phi(x)v(x,0)dx - \int_0^\infty \int_{-\infty}^\infty uv_t dx\, dt.$$

Then, by (1.8) and the previous calculations we get

$$0 = -\int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x]\, dx\, dt - \int_{-\infty}^\infty \phi(x)v(x,0)dx.$$

But this is true for all functions $v \in C^\infty(\mathbb{R} \times [0, +\infty))$ with compact support. Therefore, u is a weak solution of (1.7). □

The notion of weak solution allows solutions which is not necessarily continuous. However, weak solutions u have some restrictions on types of discontinuities. For example, suppose u is a weak solution of (1.5) such that u is discontinuous across some curve $x = \xi(t)$, but u is smooth on either side of the curve. Let $u^-(x,t)$ be the limit of u approaching (x,t) from the left and let $u^+(x;t)$ be the limit of u approaching (x,t) from the right. We claim that the curve $x = \xi(t)$ cannot be arbitrary, but rather there is a relation between $x = \xi(t)$, u^- and u^+.

Theorem 1.2. *If u is a weak solution of (1.7) such that u is discontinuous across the curve $x = \xi(t)$ but u is smooth on either side of $x = \xi(t)$, then u must satisfy the condition*

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = \xi'(t) \tag{1.9}$$

across the curve of discontinuity, where $u^-(x,t)$ is the limit of u approaching (x,t) from the left and $u^+(x,t)$ is the limit of u approaching (x,t) from the right.

Proof. Let us suppose that u is a weak solution of (1.7), i.e,

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x]\, dx\, dt + \int_{-\infty}^\infty \phi(x)v(x,0)dx = 0,$$

for all smooth functions $v \in C^\infty(\mathbb{R} \times [0, +\infty))$ with compact support. Let v be a smooth function such that $v(x,0) = 0$, and split the first integral into the regions Ω^- and Ω^+, where

$$\Omega^- = \{(x,t) : 0 < t < \infty, -\infty < x < \xi(t)\},$$
$$\Omega^+ = \{(x,t) : 0 < t < \infty,\ \xi(t) < x < \infty\}.$$

Therefore, using $v(x,0) = 0$ we have

$$0 = \int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x]\, dx\, dt + \int_{-\infty}^\infty \phi(x)v(x,0)dx$$
$$= \iint_{\Omega^-} [uv_t + f(u)v_x]\, dx\, dt + \iint_{\Omega^+} [uv_t + f(u)v_x]\, dx\, dt. \tag{1.10}$$

Combining the Divergence Theorem [14] with the fact that v has compact support and $v(x,0)=0$ we get that

$$\iint_{\Omega^-}[uv_t+f(u)v_x]\,dx\,dt=-\iint_{\Omega^-}[u_tv+[f(u)]_xv]\,dx\,dt+\int_{x=\xi(t)}[u^-v\nu_2+f(u^-)v\nu_1]ds, \tag{1.11}$$

where $\nu=(\nu_1,\nu_2)$ is the outward unit normal to Ω^-. Similarly, we see that

$$\iint_{\Omega^+}[uv_t+f(u)v_x]\,dx\,dt=-\iint_{\Omega^+}[u_tv+[f(u)]_xv]\,dx\,dt-\int_{x=\xi(t)}[u^+v\nu_2+f(u^+)v\nu_1]ds. \tag{1.12}$$

Note that, by assumption, as u is weak solution of $u_t+[f(u)]_x=0$ and is smooth on either side of $x=\xi(t)$, therefore u is a strong solution on either side of the curve of discontinuity, i.e., we have that

$$\iint_{\Omega^+}[u_tv+[f(u)]_xv]\,dx\,dt=0=\iint_{\Omega^-}[u_tv+[f(u)]_xv]\,dx\,dt.$$

Hence, by this fact with (1.10), (1.11) and (1.12), we obtain

$$\int_{x=\xi(t)}[u^-v\nu_2+f(u^-)v\nu_1]ds-\int_{x=\xi(t)}[u^+v\nu_2+f(u^+)v\nu_1]ds=0.$$

Since this is true for all smooth functions v, we have

$$u^-v\nu_2+f(u^-)v\nu_1=u^+v\nu_2+f(u^+)v\nu_1\,,$$

which implies

$$\frac{f(u^-)-f(u^+)}{u^--u^+}=-\frac{\nu_2}{\nu_1}\,.$$

Now we know that if the curve $x=\xi(t)$ have normal vector (ν_1,ν_2), then the tangent vector is $(-\nu_2,\nu_1)$. Hence, by the choice of regions Ω^- and Ω^+, the derivative of $x=\xi(t)$ satisfies that $\frac{dt}{dx}=\frac{1}{\xi'(t)}=\frac{\nu_1}{-\nu_2}$. So, we can conclude. □

For a simples notation, we define

$$\begin{aligned}[u]&=u^--u^+,\\ [f(u)]&=f(u^-)-f(u^+),\\ \sigma&=\xi'(t).\end{aligned}$$

We call $[u]$ and $[f(u)]$ the jumps of u and $f(u)$ across the discontinuity curve and σ the speed of the curve of discontinuity. Therefore, if u is a weak solution with discontinuity along a curve $x=\xi(t)$, the solution must satisfy

$$[f(u)]=\sigma[u],\qquad \text{where } \sigma=\xi'(t). \tag{1.13}$$

This is called the *Rankine-Hugoniot jump condition*.

Example 1.1. We consider the Burgers equation with $f(u) = \dfrac{u^2}{2}$. Thus

$$\begin{cases} u_t + uu_x = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{1.14}$$

where the initial condition satisfies

$$\phi(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Solving this equation using the method of characteristics, our characteristic equations are given by

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{dx}{ds} = z \\ \dfrac{dz}{ds} = 0 \end{cases}$$

with initials conditions

$$\begin{cases} t(r,0) = 0 \\ x(r,0) = r \\ z(r,0) = \phi(r). \end{cases}$$

We see that the solutions of each equation above using this initials conditions are

$$\begin{aligned} t &= s \\ x &= \phi(r)s + r \\ z &= \phi(r). \end{aligned}$$

From these solutions, we arrive at an implicit solution for the PDE which is given by

$$u = \phi(x - ut).$$

Using the fact that $\frac{dz}{ds} = 0$, we see that u is constant along the projected characteristic curves, $x = \phi(r)t + r$. Then, for $r < 0$, $\phi(r) = 1$ and therefore, these projected curves are given by $x = t + r$ for $-\infty < r < 0$, and $u(x,t) = 1$ along these curves, this is, $u(x,t) = 1$ for $-\infty < x - t < 0$. For $0 < r < 1$, $\phi(r) = 1 - r$ then these projected curves are given by $x = (1 - r)t + r$ for $0 < r < 1$. Moreover, along these curves $u(x,t) = z(r,s) = 1 - r = \frac{1-x}{1-t}$ for $0 < \frac{x-t}{1-t} < 1$. Finally, for $r > 1$, $\phi(r) = 0$. Therefore, the projected characteristic curves are given by $x = r$ for $r > 1$, and $u = 0$ along these curves.

So, for $t \leq 1$, our solution is

$$u(x,t) = \begin{cases} 1, & x < t, \\ \frac{1-x}{1-t}, & t < x < 1, \\ 0, & x > 1. \end{cases}$$

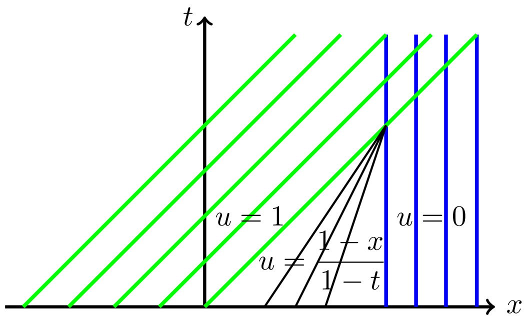

Figure 1.3. Characteristic curves for $t \leq 1$ to the problem (1.14) and some continuations for some curves.

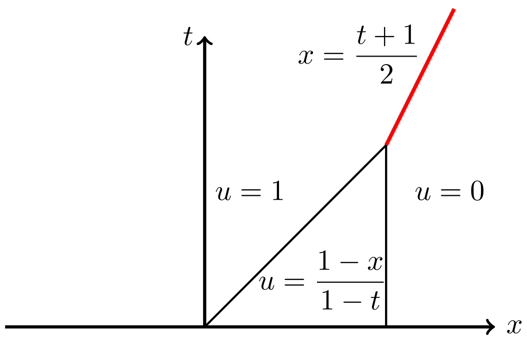

Figure 1.4. General solution for the equation (1.14) where the red line is its shock curve.

However, note that the characteristic curves intersect at $t = 1$. Beyond that time t, the different projected characteristics are asking for our solution u to satisfy different conditions, which cannot happen, and we no longer have a classical solution. From the Theorem 1.2, given that for $t \geq 1$ a weak solution of the PDE (1.14) with $\phi(x)$ satisfy the Rankine-Hugoniot jump condition (1.13). That is,

$$[f(u)] = \sigma[u] \Leftrightarrow \frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \xi'(t)(u^- - u^+)$$

Then taking into account the solution for $u(x,t)$ for $t \leq 1$, we want a curve $x = \xi(t)$ such that it contains the point $(x,t) = (1,1)$ and $u = 1$ to the left of the curve and $u = 0$ to the right of the curve, this is, $u^- = 1$ and $u^+ = 0$. In particular we have

$$\xi' = \frac{1}{2}, \qquad \text{with } (x,t) = (1,1).$$

Therefore, our curve must be given by $(x-1) = \frac{1}{2}(t-1)$ or $x = \frac{t+1}{2}$. So, for $t \geq 1$, the other part of the weak solution is

$$u(x,t) = \begin{cases} 1, & x < \frac{t+1}{2}, \\ 0, & x > \frac{t+1}{2}. \end{cases}$$

Example 1.2. We consider the Burgers equation (1.14) again, with another initial condition. So

$$\begin{cases} u_t + uu_x = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

and for $t = 0$, we have

$$\phi(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

As before, u is constant along the projected characteristic curves given by $x = \phi(r)t + r$. If $r < 0$, then $\phi(r) = 1$ which implies the projected characteristic curves are $x = t + r$ for $r < 0$ and the solution u should equal 1 along those curves. But, also, for $r > 0$, $\phi(r) = 0$ which means the projected characteristic curves are given by $x = r$ for $r > 0$ and the solution u should equal 0 along these curves. This curves have intersection at $t = 0$.

Clearly, in this case we can not hope to find any continuous solution which solves this problem. Again, we look for a weak solution, by looking for a piecewise continuously differentiable function which satisfies the Rankine Hugoniot jump condition (1.13). We want to find a curve $x = \xi(t)$ such that $u^- = 1$ to the left of the curve and $u^+ = 0$ to the right of the curve. Hence, we have

$$\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \xi'(t)(u^- - u^+) \Leftrightarrow \xi' = \frac{1}{2}, \quad \text{with } (x,t) = (0,0).$$

The weak solution of system (1.14) taking into account the curve solution above $x = \frac{x}{2}$ is

$$u(x,t) = \begin{cases} 1, & x < \frac{t}{2}, \\ 0, & x > \frac{t}{2}. \end{cases}$$

Example 1.3. We consider the Burgers equation (1.14) once again,

$$\begin{cases} u_t + uu_x = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

but now impose the initial condition

$$\phi(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Looking at our characteristics, we see that u should be constant along the projected characteristic curves, $x = \phi(r)t + r$. Here, if $r < 0$, then $\phi(r) = 0$ and therefore, $x = r$. If $r > 0$, then $\phi(r) = 1$ and therefore, $x = t + r$. Consequently, we have no crossing of characteristics. However, we still have a problem. since we must define our solution in the region on which we do not have enough information.

Let us consider the follow functions $u_1(x,t)$ and $u_2(x,t)$ given by

$$u_1(x,t) = \begin{cases} 0, & x < \frac{t}{2}, \\ 1, & x > \frac{t}{2}, \end{cases}$$

$$u_2(x,t) = \begin{cases} 1, & x \le 0, \\ \frac{x}{t}, & 0 < x \le t, \\ 0, & x \ge 1. \end{cases}$$

Clearly, $u_1(x,t)$ is a classical solution on either side of the curve of discontinuity $x = t/2$. In addition, from the work of the previous example, it is easy to see that $u_1(x,t)$ satisfies the Rankine-Hugoniot jump condition along the curve of discontinuity. Therefore, $u_1(x,t)$ is a weak solution of (1.14) with initial condition $\phi(x)$ above.

Note that $u_2(x,t)$ is a continuous solution of (1.14) with respect to $\phi(x)$. This type of solution which fans the wedge $0 < x < t$ is called a *rarefaction wave*.

1.2.3 Entropy Condition

Let us consider a quasilinear equation mentioned in (1.5) of the form

$$u_t + [f(u)]_x = 0. \tag{1.15}$$

This equation can also be written in the form

$$u_t + f'(u)u_x = 0.$$

The characteristic equations associated to the PDE above are given by

$$\begin{cases} \dfrac{dx}{ds} = f'(z) \\ \dfrac{dt}{ds} = 1 \\ \dfrac{dz}{ds} = 0 \end{cases}$$

From these equations, we see that the speed of a solution u is given by

$$\frac{dx}{dt} = \frac{\frac{dx}{ds}}{\frac{dt}{ds}} = f'(u).$$

For an equation of the form (1.15), we only allow for a curve of discontinuity in our solution $u(x,t)$ if the wave to the left is moving faster than the wave to the right. That is, we only allow for a curve of discontinuity between u^- and u^+ if

$$f'(u^-) > \sigma > f'(u^+)\,. \tag{1.16}$$

This is called the *entropy condition*. We say that a curve of discontinuity is a *shock curve* for a solution u if the curve satisfies the Rankine-Hugoniot jump condition (1.13) and the entropy condition (1.16). Therefore, to eliminate the physically less realistic solutions, we only accept solutions u for which curves of discontinuity in the solution are shock curves. We state this more precisely as follows. Consider the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{1.17}$$

We say u is a *weak admissible solution* of (1.17) if u is a weak solution such that any curve of discontinuity for u is a shock curve.

Example 1.4. In general, in the previous examples we have that $f(u) = \frac{u^2}{2}$ then $f'(u) = u$. In the Examples 1.1 and 1.2, $\xi' = 1/2$, $u^- = 1$ and $u^+ = 0$ so we see that the curve $\xi = x = t/2$ satisfy the entropy condition (1.16) hence $x = t/2$ is a shock curve.

Moreover, in the Example 1.3, $\xi' = 1/2$, $u_1^- = 1$ and $u_1^+ = 0$, thus

$$f'(u^-) = u^- = 0 \ngtr \frac{1}{2} \ngtr f'(u^+) = u^+ = 1.$$

Therefore, u_1 does not satisfy the entropy condition along the curve of discontinuity $x = t/2$. Consequently, $x = t/2$ is not a shock curve, and, therefore, u_1 is not an admissible solution. The solution u_2, however, is a continuous solution. Therefore, we accept this solution as the physically relevant one.

Example 1.5. We consider the Burgers equation with $f(u) = u(1-u)$,

$$\begin{cases} u_t + (1-2u)u_x = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = u_0(x) \;=\; u^{\lambda,\rho}(x), & x \in \mathbb{R}, \end{cases} \tag{1.18}$$

and the initial condition is

$$u^{\lambda,\rho}(x) \;:=\; \begin{cases} \lambda, & \text{if } x \le 0 \\ \rho, & \text{if } x > 0, \end{cases} \quad \text{where } \lambda, \rho \in [0,1]. \tag{1.19}$$

This problem is called the *Riemann problem*. There exists a more general equation, called the *viscid Burgers equation*, defined as follows

$$\frac{\partial u}{\partial t} = -\frac{\partial u(1-u)}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2}, \tag{1.20}$$

where $\beta > 0$, but to our goal is to solve the PDE (1.18). Following the characteristic method we obtain the system

$$\begin{cases} \dfrac{dt}{ds} = 1, \\ \dfrac{dx}{ds} = 1 - 2z, \\ \dfrac{dz}{ds} = 0, \end{cases}$$

with initial conditions

$$\begin{cases} t(r,0) = 0, \\ x(r,0) = r, \\ z(r,0) = u^{\lambda,\rho}(r). \end{cases}$$

Then we have that the solution is given by

$$\begin{cases} t = s, \\ x = (1-2z)s + r, \\ z = u^{\lambda,\rho}(r), \end{cases}$$

and we can conclude that the implicit solution of (1.18) is

$$u = u^{\lambda,\rho}(x - (1-2u)t).$$

Using the fact that $\frac{dz}{ds} = 0$, we see that u is constant along the projected characteristic curves, $x = (1 - 2u^{\lambda,\rho}(r))s + r$. We see that for $r \leq 0$, $u^{\lambda,\rho}(r) = \lambda$ which implies the projected characteristic curves are $x = (1-2\lambda)t + r$ for $r \leq 0$ and the solution u should equal λ along those curves. Also, for $r > 0$, $u^{\lambda,\rho}(r) = \rho$ which means the projected characteristic curves are given by $x = (1-2\rho)t + r$ for $r > 0$ and the solution u should equal ρ along these curves. Then, having into account that $u^- = \lambda$, $u^+ = \rho$ and $f'(u) = 1 - 2u$ by the mentioned above, we have that $f'(u^-) = 1 - 2\lambda$ and $f'(u^-) = 1 - 2\lambda$. Therefore

$$\sigma = \xi'(t) = \frac{\lambda(1-\lambda) - \rho(1-\rho)}{\lambda - \rho} = 1 - \lambda - \rho.$$

Hence we have the shock case when are satisfied (1.13) and (1.16). By construction of ξ (1.13) follow and the entropy condition (1.16) is given by

$$f'(u^-) = \rho^2 - \rho > \sigma > f'(u^+) \leftrightarrow 1 - 2\lambda > 1 - \lambda - \rho > 1 - 2\rho,$$

which is valid when $\lambda < \rho$.

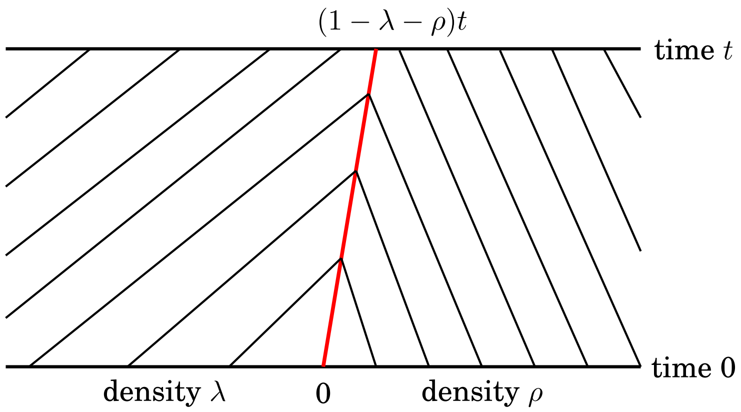

Figure 1.5. Shock case $\rho > \lambda$

We can conclude that the characteristics starting in r and $-r$ go at velocity $1-2\rho$ and $1-2\lambda$, respectively, and that also the line shock travels at velocity $1-\rho-\lambda$. Therefore, a solution of the PDE (1.18) in this case is

$$u(x,t) = u^{\lambda,\rho}(x - (1-\lambda-\rho)t)(x,t) = \begin{cases} \lambda, & \text{if } x \leq (1-\lambda-\rho)t, \\ \rho, & \text{if } x > (1-\lambda-\rho)t. \end{cases} \tag{1.21}$$

In the case $\lambda > \rho$ the characteristics emanating at the left of the origin have speed $(1-2\lambda) < (1-2\rho)$. In this case that there are many weak solutions in the sense of (1.6). Two of this solutions are the rarefaction fan or transonic rarefaction and rarefaction shock show in the Figure 1.6 and 1.7.

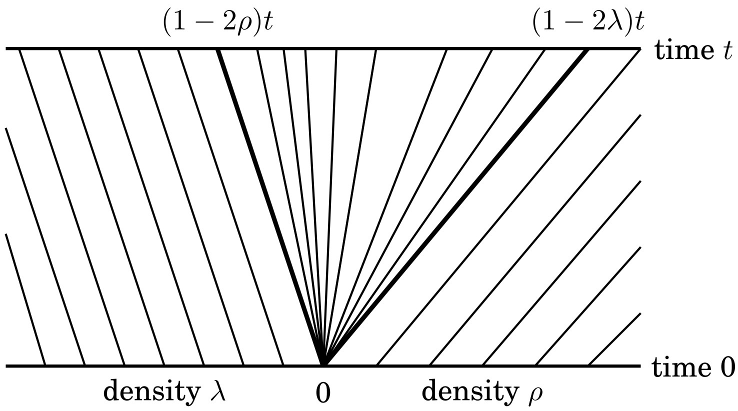

Figure 1.6. Rarefaction fan $\lambda > \rho$

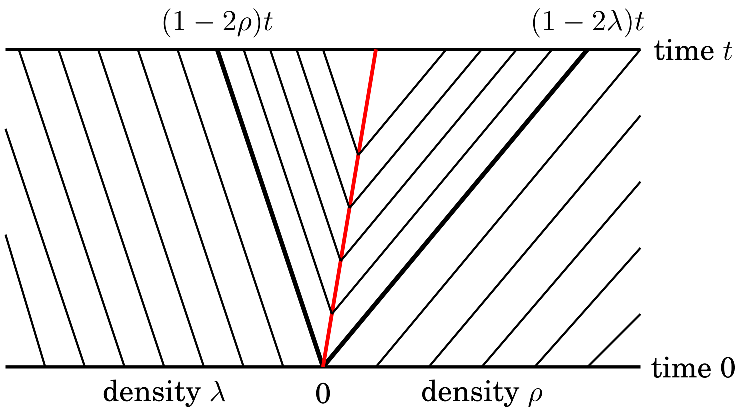

Figure 1.7. Rarefaction shock $\lambda > \rho$

For these many weak solutions, only one of them is “physical”, i.e., is the unique solution for the viscid Burgers equation in (1.20), taking $\beta \to 0$. This unique solution is the rarefaction fan solution. Note that the family of characteristics emanating from the origin travels with speeds $(1-2\alpha)$ for $\lambda \geq \alpha \geq \rho$. In the left side of the characteristic line which travels with velocity $(1-2\lambda)$ is constant equal to λ, also happens on the right side of the characteristic with speed $(1-2\rho)$ is equal to ρ. Finally, we have u is constant in the characteristic with speed $(1-2\alpha)$, this is $u((1-2\alpha)t, t) = \alpha$, observe that the equation of the characteristic line is $x = (1-2\alpha)t$ iff $\alpha = \dfrac{x-t}{-2t}$.

Hence, the solution of the rarefaction case is

$$u(x,t) = \begin{cases} \lambda, & \text{if } x \leq (1-2\lambda)t, \\ \dfrac{t-r}{2t}, & \text{if } (1-2\lambda)t \leq x \leq (1-2\rho)t, \\ \rho, & \text{if } x > (1-2\rho)t. \end{cases} \tag{1.22}$$

1.3 Reversibility and Burke's theorem

1.3.1 Reversibility and reverse process

In general, X_t represents a continuous time Markov chain with enumerable space stated E and rates

$$q(x,y) \;=\; \lim_{h\to 0}\frac{\mathbb{P}(X_{t+h}=y|X_t=x)}{h}.$$

We say that $\pi : E \to [0,1]$ is a *probability measure reversible* for the process X_t if $\forall x,y \in E$

$$\pi(x)q(x,y) = \pi(y)q(y,x)\,, \tag{1.23}$$

$$\sum_x \pi(x) = 1. \tag{1.24}$$

In other words, if the process X_t has π as initial reversible measure then the probability of look a jump in the direction $x \to y$ in a small time is the same that observe of $y \to x$. For the next propositions we will omit its demonstrations.

Proposition 1.1. *If π is reversible for X_t, then π is invariant for X_t.*

Proposition 1.2. *The Markov process X_t accept π as reversible measure iff π is a probability and $\forall t \geq 0$*

$$\pi(x)P_t(x,y) = \pi(y)P_t(y,x), \tag{1.25}$$

where $P_t(x,y)$ is the transition probability in the time t.

The next proposition establishes that a trajectory, on a reversible measure, have the same law that a trajectory looking from back for front.

Proposition 1.3. *Lets $0 = t_1 < t_2 < ... < t_n < t$ and $x_1, x_2, ..., x_n \in E$. For $s_i = t - t_{n-i}$ we have*

$$\mathbb{P}_\pi(X_{t_1} = x_1, ..., X_{t_n} = x_n) = \mathbb{P}_\pi(X_{s_1} = x_n, ..., X_{s_n} = x_1),$$

where $\mathbb{P}_\pi$ represents the distribution of the process starting at the stationary measure π.

We supose that that X_t its defined $\forall t \in \mathbb{R}$ and X_t is in equilibrium from $-\infty$, i.e., $\mathbb{P}_\pi(X_t = x) = \pi(x)\ \forall t \in \mathbb{R}$. Therefore, the Markov *reverse process* of X_t is

$$X_t^* \;:=\; \lim_{s\uparrow t} X_{-s} \tag{1.26}$$

Note that the fact of X_t^* to be Markovian follow of 1.3. Also, we can observe that X_t^* have the same jumps that X_t and the definition using the left limit ensures that the trajectories are right continuous. We can see in the Figure 1.8 a realization of the process X_t in black and in red the reverse process of this. Note that if the graph of $X_t(\omega)$ is symmetric with respect to axis y then the reverse process coincides with the original process, i.e., $X_t(\omega) = X_t^*(\omega)$ for all t.

Proposition 1.4. *Let X_t a process with invariant measure π. If X_t^* the reverse process with relation to π then*

1. *π is invariant for X_t^**
2. *X_t^* has the same distribution that X_t in equilibrium iff π is reversible for X_t.*

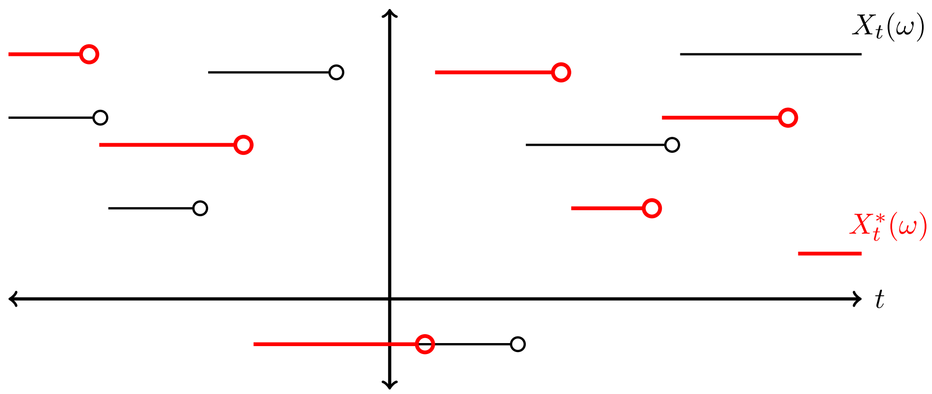

Figure 1.8. Ralization of the process continuous X_t with spaces states in $\mathbb{R}$ and their respective reverse process.

1.3.2 Process $(M|M|1)$ and Burke's theorem

The line $(M|M|1)$ is a Markov process X_t with $E = \mathbb{N}$ and their rates are

$$q(x,y) = \begin{cases} \lambda, & \text{if } y = x+1, \\ \mu, & \text{if } x \geq 1 \text{ and } y = x-1, \\ 0, & \text{other wise.} \end{cases} \tag{1.27}$$

The random variable X_t is a number of persons waiting in line to be attended, with the attention is by order of arrival, one person at a time and are attended immediately. The first letter M in this process symbolizes that the arrives are Markovian, in this case a Poisson process with parameter λ, the second M also with respect to the services are exponential of parameter μ. Finally the "1" is only a part serving.

The next part is find a invariant measure, in this case is it valid to ask: there exist a reversible measure? An affirmative answer follows by the Proposition 1.1. Taking in account (1.23) and (1.27) we have $\pi(x)\lambda = \pi(x+1)\mu$, recursively we obtain $\pi(x) = \left(\frac{\lambda}{\mu}\right)^x \pi(0)$, if $x \geq 1$. So, using the second condition of reversibility (1.24)

$$1 = \sum_{x\geq 0} \pi(x) = \pi(0) + \sum_{x \geq 1} \pi(x) = \pi(0) + \pi(0) \sum_{x\geq 1} \left(\frac{\lambda}{\mu}\right)^x \overset{\text{if } \lambda < \mu}{=} \pi(0) \left(\frac{\mu}{\mu - \lambda}\right).$$

Hence, we can conclude that for $x \geq 0$, if $\lambda < \mu$,

$$\pi(x) = \left(\frac{\lambda}{\mu}\right)^x \left(1 - \frac{\lambda}{\mu}\right). \tag{1.28}$$

Let us suppose that X_t is defined for $t \in \mathbb{R}$ and this one is equilibrium, i.e., $\mathbb{P}(X_t = x) = \pi(x)\ \forall t$. We define the process of arrivals and departures, A_t and D_t, respectively as

$$A_t - A_s = \sum_{u \in [s,t]} \mathbb{1}_{\{X_u - X_{u^-} = 1\}},$$

$$D_t - D_s = \sum_{u \in [s,t]} \mathbb{1}_{\{X_u - X_{u^-} = -1\}}.$$

Considering the above mentioned we can enunciate the following.

Theorem 1.3. *(Burke's Theorem (M | M | 1)) X_t with stationary distribution* (1.28)*, then D_t is a Poisson process with parameter λ.*

Proof. Let be A_t^* the arrivals reverse process given by

$$A_t^* - A_s^* = \sum_{u \in [s,t]} \mathbb{1}_{\left\{X_u^* - X_{u^-}^* = 1\right\}}.$$

Similarly, the departures reverse process is

$$D_t^* - D_s^* = \sum_{u \in [s,t]} \mathbb{1}_{\left\{X_u^* - X_{u^-}^* = -1\right\}}.$$

Using the definition of reverse Process (1.26) and the above definitions we have that $A_t - A_s = D_{-s}^* - D_{-t}^*$ and $D_t - D_s = A_{-s}^* - A_{-t}^*$. But since π in (1.28) is reversible for X_t, implies by the proposition 1.4 that X_t^* have the same distribution of X_t so, given that A_t is Poisson of parameter λ, A_t^* is a Poisson process with parameter λ. Therefore the theorem follow taking in account the inequality of the arrives reverse process and the departure process. □

Chapter 2

The Hydrodynamic Limit

An important subject in Statistical Physics is the comprehension of the hydrodynamic behavior of interacting particle systems. Roughly speaking, given a discrete system that evolves in time, its hydrodynamic limit consists on the limit for the time trajectory of the spatial density of particles (as some parameters are rescaled, in general, space and time). Proving rigorously such scaling limit is often a mathematical problem of deep technical difficulty.

2.1 Heuristic derivation of Burgers equation from TASEP

Our goal is to relate the density of TASEP and the solution of the Burgers equation. To do so let us see first an intuitive way. Using the forwards Kolmogorov equation for the function $f(\eta) = \eta(x)$ we get

$$\frac{d}{dt}\mathbb{E}[\eta_t(x)] = \mathbb{E}[-\eta_t(x)(1-\eta_t(x+1)) + \eta_t(x-1)(1-\eta_t(x))]. \tag{2.1}$$

We fix $\varepsilon > 0$, which will go later to zero, and define

$$u^\varepsilon(r,t) := \mathbb{E}[\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)].$$

Note that the part $\varepsilon^{-1}r$ should be integer by definition of $\eta_{\varepsilon^{-1}t}$ then we abuse of notation writing $\varepsilon^{-1}t$ instead of your integer part. Considering (2.1), using the definition above and the chain rule in this equation we have

$$\frac{d}{dt}u^\varepsilon(r,t) = \varepsilon^{-1}\mathbb{E}[-\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)(1-\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r+1))+\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r-1)(1-\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r))]. \tag{2.2}$$

Assume that there exist the limit $u(r,t) := \lim\limits_{\varepsilon\to 0} u^\varepsilon(r,t)$ and the distribution of $\eta_{\varepsilon^{-1}t}$ around $\varepsilon^{-1}r$ is approximately product, that is,

$$\lim_{\varepsilon\to 0}\mathbb{E}[\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r+1)] \;=\; (u(r,t))^2.$$

Assume further that $u(r,t)$ is differentiable in r. Using all this in (2.2) we have that the right hand side must converge to minus the derivative with respect to r of $u(r,t)(1-$

$u(r,t))$. Hence the limiting $u(r,t)$ must satisfy the Burgers equation. This heuristic argument may also be a script of a proof of the convergence of the TASEP density to a solution of the Burgers equation, but we show this convergence in other terms which we will describe next.

2.2 Hydrodynamic limit

In this section we study the hydrodynamic limit, for this, we will take into account two concepts. The variations between a system and the exterior are controlled by intensive parameters (do not depend on the mass), we say that the system is in *global equilibrium* if such parameters are homogeneous in the system, while in the *local equilibrium* these vary on the space and the time, but very slowly, so that we can assume the equilibrium on a neighborhood of any point.

2.2.1 General case

Consider the Burgers equation given in (1.5) with $f(u) = u(1-u)$ and initial condition u_0 such that there exist a unique entropic weak solution $u(r,t)$ for the above initial value problem, but first let us see some definitions. Recall

$$\begin{aligned} U &= \{U(x) : x \in \mathbb{Z}\} \\ &:= \text{be a collection of iid random variables uniformly distributed in } [0,1]. \end{aligned} \tag{2.3}$$

Taking into account the initial condition u_0 we can define

$$\zeta^\varepsilon(x) := \mathbb{1}_{\{U(x)\leq u_0(\varepsilon x)\}}. \tag{2.4}$$

This is, for each $\varepsilon > 0$, the random configuration ζ^ε is a sequence of independent Bernoulli random variables whose parameter is induced by u_0 for the mesh ε. Therefore, the TASEP with random initial configuration ζ^ε is given by:

$$\zeta_t^\varepsilon := \eta_t[\zeta^\varepsilon, \omega]. \tag{2.5}$$

Let $z \in \mathbb{R}$ the translation operator, denoted by τ_z, is defined by $(\tau_z\eta)(x) = \eta(\lfloor x+z \rfloor)$ where $\lfloor z \rfloor$ is the integer part of z.

Theorem 1. *Let $u(r,t)$ be the solution of the Burgers equation with initial condition u_0. Let ζ^ϵ be given by (2.4) and ζ_t^ϵ be the TASEP with initial configuration ζ^ϵ defined in (2.5).*

Then, it happens that
Convergence of the density fields: *For all real numbers $a < b$ and for all $t \geq 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x: a\leq x\leq b} \zeta^\varepsilon_{\varepsilon^{-1}t}(x) = \int_a^b u(r,t)dr, \quad a.s. \tag{2.6}$$

Local equilibrium: *At the continuity points of $u(r,t)$ we have that*

$$\lim_{\varepsilon\to 0} \mathbb{E}[f_A(\tau_{\varepsilon^{-1}r}\zeta^\varepsilon_{\varepsilon^{-1}t})] = u(r,t)^{|A|}. \tag{2.7}$$

In the above theorem we are ignoring the integer parts by simplicity. We recall that $f_A(\eta)$ is the function which is 1 if $\eta(x) = 1, \forall x \in A$ and 0 in other case. When $A = 0$, the limit (2.7), is called *density profile* and has the form

$$\lim_{\varepsilon \to 0} \mathbb{E}[\zeta^{\varepsilon}_{\varepsilon^{-1}t}(\varepsilon^{-1}r)] = u(r,t). \tag{2.8}$$

In both equations (2.6) and (2.7) the random variable $\zeta^{\varepsilon}_{\varepsilon^{-1}t}$ is choosing values more finely with respect $u_0(\varepsilon x)$ and a big time $\varepsilon^{-1}t$ as it ε grows, also in (2.7) with this growth the translation $\varepsilon^{-1}r$ causes that the random variable take values far left in comparison to the original. The convergence of density fields tell us that the Riemann sum of the above random variable on the interval $[a, b]$, with length of partition ε, is well approximated by the integral of the density in such interval in a given time. While local equilibrium says that locally the space marginal of this process, in a finite box around a point, converges to the measure product with parameter given by the density in the point. Here we call it local equilibrium because the equality holds for a small region of the space and for changes in the variables that are actually not infinitely slow.

2.2.2 Shock and rarefaction fan cases

We consider the case corresponding to $t = 1$ and $u_0 = u^{\lambda,\rho}$ in the Burgers equation (1.18). Taking into account $\lambda, \rho \in [0, 1]$ and U in (2.3) we obtain the initial configuration $\eta^{\lambda,\rho} = \eta^{\lambda,\rho}[U]$ given by

$$\eta^{\lambda,\rho}(x) := \begin{cases} \mathbb{1}_{\{U(x)\le\lambda\}}, & \text{if } x \le 0 \\ \mathbb{1}_{\{U(x)\le\rho\}}, & \text{if } x > 0, \end{cases} \tag{2.9}$$

with this we can obtain the TASEP previously built

$$\eta^{\lambda,\rho}_t := \eta_t[\eta^{\lambda,\rho}, \omega],$$

which is a function of U and ω. The following theorem is a particular case of the Theorem 1, but we prove it using special definitions and methods.

Theorem 2. *For all real numbers $a < b$,*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{x:at\le x\le bt} \eta^{\lambda,\rho}_t(x) = \int_a^b u(r,1)dr, \quad a.s. \tag{2.10}$$

And, the continuity points of $u(\cdot, 1)$, we have

$$\lim_{t\to\infty} \mathbb{E}[f_A(\tau_{tr}\eta^{\lambda,\rho}_t)] = u(r,1)^{|A|}. \tag{2.11}$$

Note that a difference with respect the Theorem 1 is that, we fix a macroscopic time equal to 1 and use t as scaling parameter also in the equation (2.10) we do not use the scaling mesh ζ^{ε} defined in (2.4) because this mesh is included in the index of summation.

2.3 Some definitions and results

In this part we introduce some definitions and results that will be useful for establishing a demonstration of the Theorem 2.

2.3.1 The tagged particle

Let η a configuration with infinitely many particles in $\mathbb{Z}$. We can enumerate and tag each particles as follows: the particle zero is the first particle that appear to the left of zero and the other are labeled recursively, with index in $\mathbb{Z}$, as they come out on the left or right of the particle zero , i.e.,

$$X(i)[\eta] := \begin{cases} \max\{x \leq 0 : \eta(x) = 1\}, & \text{if } i = 0 \\ \min\{x > X(i-1) : \eta(x) = 1\}, & \text{if } i > 0 \\ \max\{x < X(i+1) : \eta(x) = 1\}, & \text{if } i < 0. \end{cases} \tag{2.12}$$

Clearly we have a dependence on the initial configuration. In our case, we are interested in initial configurations with a particle at the origin, so we define

$$\tilde{\eta}(x) := \begin{cases} 1, & \text{if } x = 0, \\ \eta(x), & \text{otherwise,} \end{cases} \tag{2.13}$$

too, we get the configuration at time t of $\tilde{\eta}$, $\tilde{\eta}_t := \eta_t[\tilde{\eta}, \omega]$.

We can see the graphical representation of the tagged particles and its trajectories in the Figure 2.1, so we can know the position of the particles in any time t, let be $X_t(i)[\tilde{\eta}, \omega]$ the position of the $i-$th particle at time t; when η and ω are understood we just denote $X_t(i)$. We call $X_t := X_t(0)$ the position of the tagged particle initially at the origin and define the process as seen from that tagged particle by $\tau_{X_t}\eta_t[\tilde{\eta}, \omega]$.

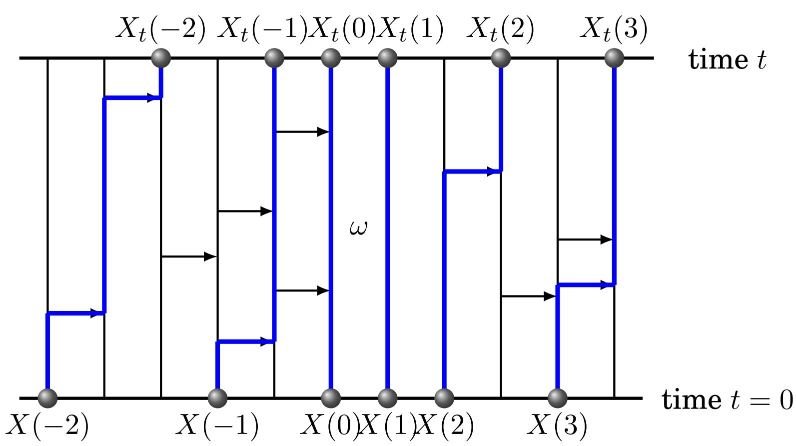

Figure 2.1. Trajectories made for the tagged particles with initial configuration $\tilde{\eta}$ and realization ω.

Now, we consider the configuration η^ρ defined in (1.1), applying (2.13) we obtain our new initial configuration $\tilde{\eta}^\rho$ which has law Bernoulli product measure conditioned to have a particle at the origin. The distribution of $\tilde{\eta}^\rho$ is invariant for the process as seen from the tagged particle, i.e., $\tau_{X_t}\tilde{\eta}^\rho_t$ has the same distribution as $\tilde{\eta}^\rho$ for all $t \geq 0$.

The next proposition is a celebrated fact, usually attributed to Kesten (see [13]. However, since we couldn't find a proof for such result in the literature, an argument is provided below.

Proposition 2.1. *Let X_t be the position of the tagged particle initially at the origin for the process with random initial configuration $\tilde{\eta}^\rho$. Then there exist a Poisson point process N_t of parameter $(1-\rho)$ such that X_t coincides with the number of arrivals of N_t.*

Proof. Since the particles to the left of the tagged particle do not interfere with the movement of the tagged particle, we will deal with particles with positive tag.

Given an initial configuration η and a realization ω we can make a bijection between the TASEP evolution for these elements and a system of infinity queues with index in $\mathbb{N} \cup \{0\}$ as follows.

The ith particle is associated with the queue i and the number of customers in the queue i is the number of empty sites between the particle i and $i+1$. The queue system evolves with respect to TASEP, i.e., if we have an arrow such that the ith particle can move to the rigth, then customer in the queue i moves to the left going to the queue $i-1$ or leaving the system when $i=0$. See an illustration of this construction in the Figure 2.2.

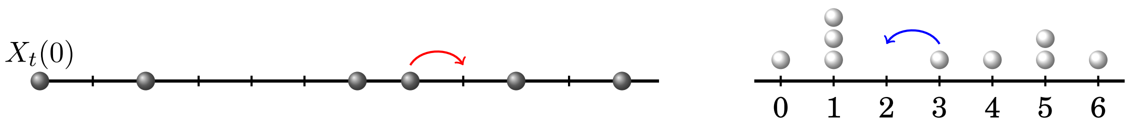

Figure 2.2. Illustration of the bijection between the TASEP and the infinity queues system.

Let $\tilde{\eta}^\rho$ the initial random configuration, which always has the tagged particle at the origin, i.e., $X_0(0)=0$. Note that the bijection implies that the number of departures of the queue 0 coincides with the position of X_t. By construction of these queues, the customers are served by means of a Poisson process of rate 1. Also, we know that the distribution of η^ρ is product Bernoulli with parameter ρ, which is invariant for the TASEP by Lemma 1.1, this implies that the queues system has a invariant product geometric measure with parameter $(1-\rho)$.

We consider the process with $N+1$ queues:

$$\eta_t^N := (\eta_t^N(0), ..., \eta_t^N(N)),$$

where each $\eta_t^N(j)$ indicates the number of customers in the jth queue. Let us assume that the customers get into the queue N via a Poisson point process of rate $(1-\rho)$, which is denoted by PPP$(1-\rho)$. Assume that this process starts from the initial (invariant) measure given by

$$\prod_{j=0}^{N} \text{Geom}(1-\rho)\,.$$

Note that if we have the same conditions by the process η_t^{N+1}, using the Burke's Theorem 1.3 in the last queue, since the attending rate 1 is greater than the arrivals rate $(1-\rho)$ and the invariant initial measure is geometric, we have that the departure process in the queue $N+1$ is a PPP$(1-\rho)$.

Denote by $D_t^N(0)$ the departure process of the queue 0 of the process η_t^N. Therefore, if we have the same initial measure and Poisson processes in every process η_t^N. Using recursively the above remark, we have that $D_t^N(0)$ is a PPP$(1-\rho)$ for any $N \in \mathbb{N}$. Moreover, under these conditions, there exists a coupling of all these processes such that the marginal process of η_t^{N+1} coincides with η_t^N, i.e.,

$$(\eta_t^{N+1}(0), ..., \eta_t^{N+1}(N)) = (\eta_t^N(0), ..., \eta_t^N(N)).$$

Let be η_t^∞ the process with infinity queues as above, which is the process given by the bijection initially mentioned.

We claim that the departure process in the queue 0 by this process, which we will denote by $D_t^\infty(0)$, is a PPP$(1-\rho)$ provided the system starts from the invariant product measure $\prod_{j=0}^{\infty} \text{Geom}(1-\rho)$. Note that this claim and the aforementioned bijection immediate imply that X_t is a Poisson point process with parameter $(1-\rho)$, finishing the proof.

In order to prove the claim, consider the random variable

$$T = \inf\big\{k \in \mathbb{N} \cup \{0\} : \text{there are no connections between the queues } k \text{ and } k+1 \text{ in } [0,t_0]\big\}.$$

Using the Borel-Cantelli Lemma we have that $\mathbb{P}[T<\infty]=1$. Let

$$A_k \;=\; \big[T=k\big]\,,$$

which are disjoint events and note that $[T<\infty] = \bigcup_{k=0}^{\infty} A_k$ and $\mathbb{P}\Big[\bigcup_{k=0}^{\infty} A_k\Big] = 1$. By construction,

$$(\eta_t^\infty(0), ..., \eta_t^\infty(k))\mathbb{1}_{A_k} \;=\; (\eta_t^k(0), ..., \eta_t^k(k))\mathbb{1}_{A_k}, \quad \text{for any } t \in [0,t_0].$$

Therefore, $D_t^\infty(0) = D_t^k(0)$ for all $t \in [0,t_0]$ in A_k. Since D_t^k is a PPP$(1-\rho)$ and $D_t^{k+1}(0) = D_t^k(0)$ in the time interval $t \in [0,t_0]$ for any $k \in \mathbb{N}$, we then conclude that $D_t^\infty(0)$ is a PPP$(1-\rho)$ on $[0,t_0]$. And since t_0 is arbitrary, it implies that $D_t^\infty(0)$ is a PPP$(1-\rho)$ on $[0,\infty]$, concluding the proof. □

Corollary 2.1. ***(Law of large numbers for the tagged particle)*** *Let X_t be the position of the tagged particle initially at the origin for the process with random initial configuration $\tilde{\eta}^\rho$. Then*

$$\lim_{t\to\infty} \frac{X_t}{t} \;=\; 1-\rho, \quad a.s. \tag{2.14}$$

Proof. This convergence follows from the above theorem and the strong law of large numbers for renewal processes (see [6, Theorem 4.1]), considering in our situation the renewal process X_t, where the time between arrivals are given by independent exponential variables of rate $(1-\rho)$. □

2.3.2 Coupling and first and second class particles

Taking into account the graphical construction of the TASEP introduced before, we can define naturally the *coupling*, this is done via two initial configurations η and $\eta\prime$ given by

$$\{(\eta_t, \eta'_t) : t \geq 0\} := \{(\eta_t[\eta, \omega], \eta_t[\eta\prime, \omega]) : t \geq 0\}. \tag{2.15}$$

Note that, by definition we have two ordened pairs which each component has infinite length subject to same TASEP realization ω, therefore graphically we shall see this as two randoms configuration very close evolving in time, see Figure 2.3. Hence, if we have a arrow in x to $x+1$ at time t and if some configuration has a particle in x at t^- this jump for $x+1$ submitted to the rules of TASEP.

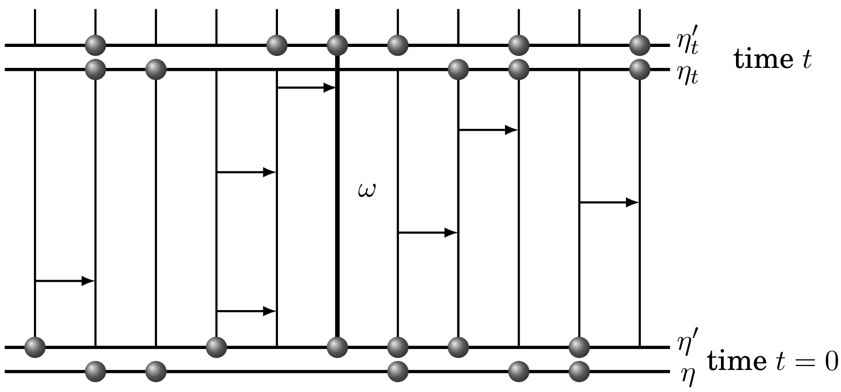

Figure 2.3. Graphical interpretation for the coupling with initial configurations η, η' and realization ω.

For two configurations η, η', we denoted $\eta \leq \eta'$ if every time that we have a particle in η we also have in η', i.e., $\eta(x) \leq \eta'(x)\, \forall x \in \mathbb{Z}$. Considering this we introduce the next lemma.

Lemma 2.1. *(Attractivity)* $\forall t \geq 0$ *if* $\eta \leq \eta'$, *then* $\eta_t \leq \eta'_t$.

(Discrepancy conservation) If $\eta \leq \eta'$ *and the number of discrepances is finite, i.e.,* $\sum_x (\eta'(x) - \eta(x)) < \infty$ *then*

$$\sum_x (\eta'(x) - \eta(x)) = \sum_x (\eta'_t(x) - \eta_t(x)), \quad \forall t \geq 0.$$

Proof. For $i)$ if $t = 0$, we have nothing to prove. Let be $t > 0$, is sufficient to prove that if $\eta_{t^-} \leq \eta'_{t^-}$ implies $\eta_t \leq \eta'_t$. Fix $x \in \mathbb{Z}$ and a realization ω, then we have the two cases

$(x,t) \notin \omega$ and $(x,t) \in \omega$. If we have the second case, as this process is Poisson in each integer line, there exist $t_1 < t$ such that $(x,s) \notin \omega \ \forall t_1 \leq s < t$, by assumption (taken $t_1 = t^-$ or closer to t if necessary) $\eta_{t_1}(x) \leq \eta'_{t_1}(x)$ and since there are no intermediary jumps follows that $\eta_t(x) \leq \eta'_t(x)$.

For the first case, we have two subcases. If $\forall t^- \leq s \leq t, (x,s) \notin \omega$ is immediate. For the other subcase, there exist $t^- \leq s < t$ such that $(x,s) \in \omega$, we use the second case in $[t^-, s]$ and after the first subcase in $[s,t]$, taking into account that there are no more than two particles almost surely. We can conclude by arbitrariness of x and ω.

For $ii)$, we consider the new configuration $\zeta = \eta' - \eta$ which by hipotese $\zeta \in \{0,1\}^{\mathbb{Z}}$ and has finite number of particles. As demonstrated by $i)$, we have that $\zeta_t = (\eta' - \eta)_t = \eta'_t - \eta_t$ and since the TASEP does not increase the number of particles only moves this, then

$$\sum_x \zeta(x) = \sum_x \zeta_t(x).$$

□

Fix $\eta \leq \eta'$ and define

$$\sigma_t := \eta_t[\eta, \omega]; \quad \xi_t := \eta_t[\eta', \omega] - \eta_t[\eta, \omega]. \tag{2.16}$$

By definition $\sigma_t \in \{0,1\}^{\mathbb{Z}}$ and by attractivity $\xi_t \in \{0,1\}^{\mathbb{Z}}$, which are called, respectively, *σ particles of first class* and *ξ particles of second class*. In resume, the first class particles are those on both configurations, while the second class particles are those that lays only the configuration η'. We know that these particles are not simultaneously in a same position. Therefore in the coupling we represented graphically, see Figure 2.4, by ⓪, ① and ② we mean that there is no particle, there is a first class particle or there is a second class particle, respectively.

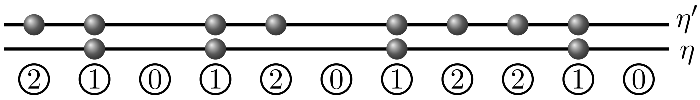

Figure 2.4. Graphical interpretation for the σ e ξ particles with initial configurations η, η' ($\eta \leq \eta'$) and a realization ω.

Now we construct a new process $\{(\sigma_t, \xi_t) : t \geq 0\}$, which will be a Markov process, this is directly function of ω and the initial configurations σ and ξ, as follows. In each integer, we know that there is just one particle σ, ξ, or there is none (denoted by 0). For the arrows between $\sigma - \sigma$, $\xi - \xi$, $\sigma - 0$ and $\xi - 0$ particles, we use the same TASEP rules, but for the arrows on $\sigma - \xi$ particles only interchange their positions if $\sigma \to \xi$. In the other case ($\xi \to \sigma$) nothing happens. In other words, the first class particles have jump preference, including jump over a second class particle, while the second class particle jumps to the right only with a hole.

The vector (σ_t, ξ_t) depends on the initial configuration $(\sigma, \xi) = (\eta, \eta' - \eta)$ and on ω. We denote the process construct above, when this needs to be stressed, by

$$(\sigma_t, \xi_t) = (\sigma_t, \xi_t)[(\sigma, \xi), \omega] = (\sigma_t[(\sigma, \xi), \omega], \xi_t[(\sigma, \xi), \omega]), \tag{2.17}$$

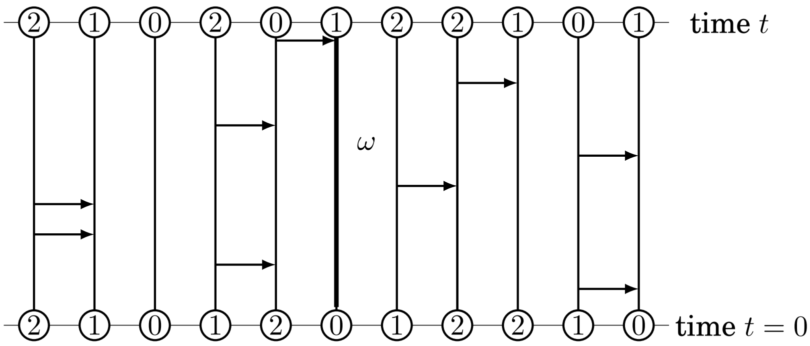

Figure 2.5. Graphical example of the behavior of the σ, ξ particles with the new rules above.

either way. Let us make some observations. First we denoted this new process by a two-dimensional vector. In this case, configurations at any time have at most one particle σ or ξ at any site. On the other hand, the new process is another way of looking at the coupling, including this is well suited with the graphical description of coupling we mentioned before.

2.3.3 Law of large numbers

Having into account a lot of the things previously addressed, we will show some versions of the law of large numbers for different elements.

Flux

Let $\{y_t\}_{t\geq 0}$ be any arbitrary trajectory in $\mathbb{R}$ with $y_0 = 0$. We define the *flux* of particles along y_t as the number of particles with no positive label that their position exceed y_t subtracting the number of particles with positive tag that their position left behind of y_t, i.e.,

$$F_{y_t}(t)[\eta, \omega] := \sum_{i\leq 0} \mathbb{1}_{\{X_t(i)[\eta,\omega] > y_t\}} - \sum_{i>0} \mathbb{1}_{\{X_t(i)[\eta,\omega]\leq y_t\}}, \tag{2.18}$$

where $X_t(i)[\eta, \omega]$ is the position of the ith particle at time t above defined. In the Figure 2.6 we see that such trajectories may not be continuous or linear.

Recalling that $\tilde{\eta}$ is the configuration η but with particle at the origin and X_t is the position of the tagged particle, then considering that $X_t(i)[\tilde{\eta}, \omega] > X_t$ iff $i \geq 1$ and $X_t(i)[\tilde{\eta}, \omega] \leq X_t$ iff $i \leq 0$ for all t and ω, we have by definition that the flux of $\tilde{\eta}$ along X_t is null, i.e.,

$$F_{X_t}(t)[\tilde{\eta}, \omega] \equiv 0. \tag{2.19}$$

However, it should be noted that the relation (2.19) is valid for any configuration η even if does not has particle at the origin.

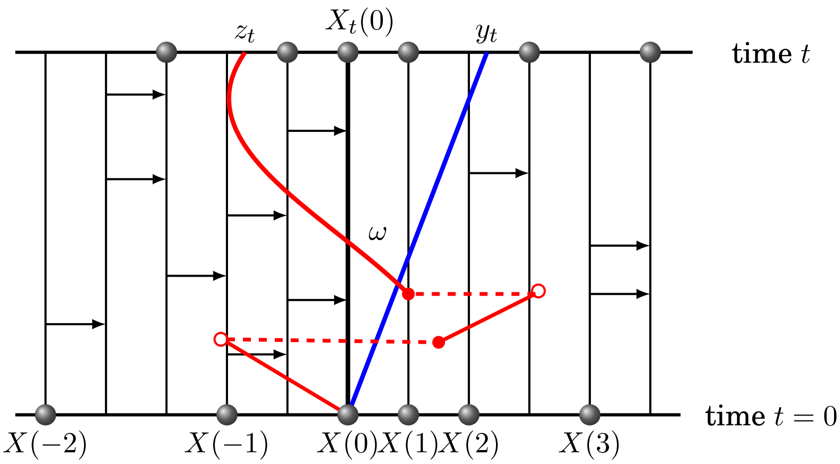

Figure 2.6. The flux along the trajectories y_t and z_t are respectively -1 and 2, for some initial configuration and realization ω

We can write in another form the flux of $\tilde{\eta}$ along y_t as follows:

$$F_{y_t}(t)[\tilde{\eta},\omega] \;=\; \sum_{x\in\mathbb{Z}} \tilde{\eta}_t(x)\big(\mathbb{1}_{\{y_t<x\le X_t\}} - \mathbb{1}_{\{X_t<x\le y_t\}}\big). \tag{2.20}$$

In this equality, by the term $\tilde{\eta}(x)$, we can replace the index i in (2.18) by x, also the first indicator function allows us take the number of values greater that y_t but with tag no positive (by $x \le X_t$), as the first sum in the definition (2.18). Similarly, the second indicator function and the second sum. These sets in both indicator functions are well defined by (2.19) because we do not count particles with the wrong label. With all this in mind, it is clear that the equalities (2.18) and (2.20) are equivalent.

Given that $\eta \le \tilde{\eta}$ has at most one discrepancy in the origin and by discrepancy conservation in the Lemma 2.1 this is conserved. Then we have

$$F_{y_t}(t)[\eta,\omega] \;:=\; \sum_{x\in\mathbb{Z}} \eta_t(x)\big(\mathbb{1}_{\{y_t<x\le X_t\}} - \mathbb{1}_{\{X_t<x\le y_t\}}\big) + h(\omega,t), \tag{2.21}$$

where $h(\omega,t) \in \{0,1\}$is a function of ω and t (in the next proposition, will be also function of U) such that $\dfrac{h(\omega,t)}{t} \underset{t\to\infty}{\to} 0$, a.s. Having this into account we have

Proposition 2.2. *Let* $a \in \mathbb{R}$. *Then,*

$$\lim_{t\to\infty} \frac{F_{at}(t)[\eta^\rho,\omega]}{t} = \rho[(1-\rho)-a], \quad a.s. \tag{2.22}$$

Proof. Given $t > 0$, we have the following cases

i) If $at \le (1-\rho)t \le X_t$ or $X_t \le (1-\rho)t \le at$, defining $\sqcup$ as the union of disjoint sets we obtain that

$$\{X_t < x \le at\} = \{X_t < x \le (1-\rho)t\} \sqcup \{(1-\rho)t < x \le at\},$$
$$\{at < x \le X_t\} = \{at < x \le (1-\rho)t\} \sqcup \{(1-\rho)t < x \le X_t\}.$$

ii) If $(1-\rho)t$ is smaller that at and X_t with $at < X_t$, then

$$\{at < x \le X_t\} = \{(1-\rho)t < x \le X_t\} \setminus \{(1-\rho)t < x \le at\},$$
$$\{X_t < x \le at\} = \{X_t < x \le (1-\rho)t\} \setminus \{at < x \le (1-\rho)t\}.$$

We have similar equalities for the cases ii') ($X_t < at$) and iii) ($(1-\rho)t$ between X_t and at) and in any case we have

$$\begin{aligned}\mathbb{1}_{\{at<x\le X_t\}} - \mathbb{1}_{\{X_t<x\le at\}} =& \big(\mathbb{1}_{\{at<x\le(1-\rho)t\}} - \mathbb{1}_{\{(1-\rho)t<x\le at\}}\big)\\ &+\big(\mathbb{1}_{\{(1-\rho)t<x\le X_t\}} - \mathbb{1}_{\{X_t<x\le(1-\rho)t\}}\big),\end{aligned}$$

using this and the equation (2.21) we have that

$$\begin{aligned}F_{y_t}(t)[\eta^\rho,\omega] =& \sum_{x\in\mathbb{Z}} \eta_t^\rho(x)\big(\mathbb{1}_{\{at<x\le(1-\rho)t\}} - \mathbb{1}_{\{(1-\rho)t<x\le at\}}\big)\\ &+ \sum_{x\in\mathbb{Z}} \eta_t^\rho(x)\big(\mathbb{1}_{\{(1-\rho)t<x\le X_t\}} - \mathbb{1}_{\{X_t<x\le(1-\rho)t\}}\big) + h(\omega,t).\end{aligned}$$

Dividing by t, in the second sum the maximum number of the numerator is the interval length, because both indicator functions take different values for all x, then the absolute value of the second term is bounded by $\frac{|X_t-(1-\rho)t|}{t}$ which converges a.s. to zero, by the Proposition 2.1, this implies that the second sum converges a.s. to zero. Given that the last term, $\frac{O(1)}{t}$, also converges a.s. to zero, then the proof will be complete if we see that the first sum converges a.s. to $\rho(1-\rho-a)$.

Let be $F(t) = \sum_{x\in\mathbb{Z}} \eta_t^\rho(x)\big(\mathbb{1}_{\{at<x\le(1-\rho)t\}} - \mathbb{1}_{\{(1-\rho)t<x\le at\}}\big)$, we assume without loss of generality that we only have the first term, this is, $a \le (1-\rho)$ this is possible because the two set of the indicator functions are disjoints. Therefore $F(t) = \sum_{at<x\le(1-\rho)t} \eta_t^\rho(x)$ and we will prove, through a three steps, that

$$\lim_{t\to\infty} \frac{F(t)}{t} = \rho(1-\rho-a) \text{ a.s.} \tag{2.23}$$

Step 1 $\lim_{k\to\infty} \frac{F(k)}{k} = \rho(1-\rho-a)$ a.s.

Using the Chebyshev inequality we have that

$$\begin{aligned}\mathbb{P}\Big[\Big|\frac{F(k)}{k} - \rho(1-\rho-a)\Big| > \varepsilon\Big] =& \mathbb{P}\Big[\Big|\frac{F(k)}{k} - \rho(1-\rho-a)\Big|^4 > \varepsilon^4\Big]\\ \le& \frac{1}{\varepsilon^4}\mathbb{E}\Big[\Big(\frac{1}{k}\sum_{ak<x\le(1-\rho)k} \eta_k^\rho(x) - \rho(1-\rho-a)\Big)^4\Big]\\ =& \frac{1}{k^4\varepsilon^4}\mathbb{E}\Big[\Big(\sum_{ak<x\le(1-\rho)k} (\eta_k^\rho(x)-\rho)\Big)^4\Big].\end{aligned} \tag{2.24}$$

By the Lemma 1.1, for any $k \ge 0$, $\eta_k^\rho(x)$ has distribution Bern(ρ) and $\eta_k^\rho(x), \eta_k^\rho(y)$ are independent for $x \ne y$. Moreover, $\eta_k^\rho(x) - \rho$ and $\eta_k^\rho(y) - \rho$ are two independent random variables with zero mean for all $x \ne y$. So

$$\mathbb{E}\big[(\eta_k^\rho(x)-\rho)(\eta_k^\rho(y)-\rho)(\eta_k^\rho(w)-\rho)(\eta_k^\rho(z)-\rho)\big] \ne 0 \text{ iff } x = y = w = z \text{ or two equal pairs.}$$

In general, if we have n summands in the sum appearing in (2.24), the quantity of those with the above property are

$$n + \binom{4}{2}\binom{n}{2} = n + \frac{3n(n-1)}{2},$$

in our case $n = \lfloor (1-\rho)k \rfloor - \lfloor ak \rfloor \leq kM$, for some M big enough, hence we get from (2.24) that

$$\begin{aligned}\mathbb{P}\Big[\Big|\frac{F(k)}{k} - \rho(1-\rho-a)\Big| > \varepsilon\Big] &\leq \frac{1}{k^4\varepsilon^4}\Big[kM\mathbb{E}\left[\eta_k(x)^4\right] + \frac{3kM(kM-1)}{2}Var(\eta_k(x))\Big] \\ &\leq \frac{ck^2}{k^4\varepsilon^4} = \frac{c}{\varepsilon^4 k^2} \quad \text{for some } c > 0.\end{aligned}$$

Hence, we have that $\sum_{k=1}^{\infty} \mathbb{P}\Big[\Big|\frac{F(k)}{k} - \rho(1-\rho-a)\Big| > \varepsilon\Big] \leq \sum_{k=1}^{\infty} \frac{c}{\varepsilon^4 k^2} < \infty$ and using the Borel-Cantelli Lemma follows the limit of the Step 1.

Step 2 $\lim_{t\to\infty} \frac{F(t)}{\lfloor t \rfloor} = \rho(1-\rho-a)$ a.s. where $t \in \mathbb{R}^+$.

Define

$$\begin{aligned}A_k &= \Big\{ \Big|\frac{F(t)}{k} - \frac{F(k)}{k}\Big| \geq \varepsilon, \text{ for some } t \in [k, k+1)\Big\} \\ &= \Big\{ \big|F(t) - F(k)\big| \geq \varepsilon k, \text{ for some } t \in [k, k+1)\Big\},\end{aligned}$$

in this case, $A_k \subseteq$ [there is at least εk arrivals of the Poisson process in the interval $[k, k+1)$]. By Burke's Theorem 1.3 we know that the number of arrivals N_k in $[k, k+1)$ is a Poisson process, we also have that, for all $k \in \mathbb{N}$, N_k are i.i.d. and since $\mathbb{E}\big[\frac{N_k}{\varepsilon}\big] = \sum_{i=1}^{\infty} \mathbb{P}\Big[\frac{N_k}{\varepsilon} \geq i\Big] < \infty$, because N_k take positive integer values, we have

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) \leq \sum_{k=1}^{\infty} \mathbb{P}\Big[N_k \geq \varepsilon k\Big] = \sum_{k=1}^{\infty} \mathbb{P}\Big[\frac{N_1}{\varepsilon} \geq k\Big] < \infty.$$

We can conclude again this step using the Borel-Cantelli Lemma and the limit in the Step 1.

Step 3 $\lim_{t\to\infty} \frac{F(t)}{t} = \rho(1-\rho-a)$ a.s. where $t \in \mathbb{R}^+$.

Once $\lim_{t\to\infty} \left(\frac{F(t)}{t} - \frac{F(t)}{\lfloor t \rfloor}\right) = 0$ a.s., adding this to the limit it the Step 2 the result follows. But this limit is due to

$$\begin{aligned}\Big|\frac{F(t)}{t} - \frac{F(t)}{\lfloor t \rfloor}\Big| &= \Big|\frac{F(t)(\lfloor t \rfloor - t)}{t\lfloor t \rfloor}\Big| \\ &= \Big|\frac{F(t)}{\lfloor t \rfloor}\Big|\frac{t - \lfloor t \rfloor}{t} \leq \Big|\frac{F(t)}{\lfloor t \rfloor}\Big|\frac{1}{t} \overset{a.s.}{\to} |\rho(1-\rho-1)| \cdot 0 = 0\end{aligned}$$

□

Tagged second class particle

Taking $0 \le \lambda < \rho \le 1$. We have that if $U(x) < \lambda$ implies $U(x) < \rho$, thus $\eta^\lambda \le \eta^\rho$. We can define now the two class process

$$(\sigma_t, \xi_t) := (\eta_t^\lambda, \eta_t^\rho - \eta_t^\lambda). \tag{2.25}$$

We can see that the marginals laws of σ_t and $\sigma_t+\xi_t$ are stationarity but the process (σ_t, ξ_t) is not stationary. Similarly to the definition of the configuration $\tilde{\eta}$ with a particle at the origin, we can define the configuration $\underset{\sim}{\eta}$ as the initial configuration with no particle at the origin, i.e.

$$\underset{\sim}{\eta}(x) := \begin{cases} 0, & \text{if } x = 0, \\ \eta(x), & \text{otherwise.} \end{cases} \tag{2.26}$$

Clearly we obtain the relation $\tilde{\eta}^\rho \ge \eta^\rho \ge \eta^\lambda \ge \underset{\sim}{\eta}^\lambda$ and we can define the process

$$(\underset{\sim}{\sigma}_t, \tilde{\xi}_t) := (\underset{\sim}{\eta}_t^\lambda, \tilde{\eta}_t^\rho - \underset{\sim}{\eta}_t^\lambda). \tag{2.27}$$

Note that the initial configuration for this process coincides with the process (σ_t, ξ_t) out of the origin while at the origin there is a second class particle since $\underset{\sim}{\sigma}_0 = 0$ and $\tilde{\xi}_0 = 1$. We state below the law of large number for the position of the tagged second class particle.

Proposition 2.3. *Take $\lambda < \rho$ and let $Y_t^{\lambda,\rho}$ be the position of the tagged particle ξ for the process (σ_t, ξ_t), initially located at the origin, that is, $Y_0^{\lambda,\rho} = 0$. Then,*

$$\lim_{t\to\infty} \frac{Y_t^{\lambda,\rho}}{t} = 1 - \lambda - \rho, \quad \textit{a.s.} \tag{2.28}$$

Proof. Denote by $G_{y_t}(t)[(\underset{\sim}{\sigma}, \tilde{\xi}), \omega]$ the flux of $\tilde{\xi}$ particles along a trajectory y_t for the process $(\underset{\sim}{\sigma}_t, \tilde{\xi}_t)$. By the definition of $\tilde{\xi}$ particles, these are the particles which are in $\tilde{\eta}^\rho$ but not in $\underset{\sim}{\eta}^\lambda$, we can write this flux as the difference of the $\tilde{\eta}^\rho$ particle flux and the $\underset{\sim}{\eta}^\lambda$ particle flux, i.e,

$$\begin{aligned} G_{y_t}(t)[(\underset{\sim}{\sigma}, \tilde{\xi}), \omega] &= F_{y_t}(t)[\tilde{\eta}^\rho, \omega] - F_{y_t}(t)[\underset{\sim}{\eta}^\lambda, \omega] \\ &= F_{y_t}(t)[\eta^\rho, \omega] - F_{y_t}(t)[\eta^\lambda, \omega] + h_2(t, \omega, U), \end{aligned}$$

where $h_2 \in \{-1, 0, 1\}$ is the error comes from (2.21). Using the law of large numbers for the flux in (2.22) for the trajectory $y_t = at$ we obtain that

$$\lim_{t\to\infty} \frac{G_{y_t}(t)[(\underset{\sim}{\sigma}, \tilde{\xi}), \omega]}{t} = [\rho(1-\rho) - \lambda(1-\lambda)] - a(\rho - \lambda), \quad \text{a.s.} \tag{2.29}$$

We can see that the limit is negative for $a > 1 - \rho - \lambda$ and positive for $a < 1 - \rho - \lambda$. Note the relationship between the flux of the $\tilde{\xi}$ particles and the position of the tagged particle given by

$$\left[G_{at}(t)[(\underset{\sim}{\sigma}, \tilde{\xi}), \omega] \ge \sum_{\substack{at \le x \le (a+c)t \\ x \in \mathbb{Z}}} \xi_t(x) \right] = \left[Y_t^{\lambda,\rho} \ge at + ct \right], \quad \text{for some } c > 0. \tag{2.30}$$

Or equivalently,

$$\left[\frac{G_{at}(t)[(\sigma,\tilde{\xi}),\omega]}{t} \geq \frac{1}{t}\sum_{\substack{at\leq x\leq (a+c)t\\ x\in\mathbb{Z}}}\xi_t(x)\right] = \left[\frac{Y_t^{\lambda,\rho}}{t} \geq a+c\right], \quad \text{for some } c>0.$$

Note that, the second term of the first set is no negative and since that the limit of the first term is positive when $a < 1-\rho-\lambda$, then we can have this assumption on a. Picking $0<\varepsilon<1-\rho-\lambda$ and choosing $c = 1-\rho-\lambda-a-\varepsilon > 0$, we have that

$$\left[\frac{G_{at}(t)[(\sigma,\tilde{\xi}),\omega]}{t} \geq \frac{1}{t}\sum_{\substack{at\leq x\leq (1-\rho-\lambda-\varepsilon)t\\ x\in\mathbb{Z}}}\xi_t(x)\right] = \left[\frac{Y_t^{\lambda,\rho}}{t} \geq 1-\lambda-\rho-\varepsilon\right].$$

Having into account that the probability for the ξ particles is $\rho-\lambda$ and taking the $\liminf$ on both sets we obtain by the limits in (2.29) and (2.23) that

$$\begin{aligned}\Big[\rho(1-\rho)-\lambda(1-\lambda)-a(\rho-\lambda) \geq (\rho-\lambda)(1-\rho-\lambda-a-\varepsilon)\Big]&\\ = \left[\liminf_{t\to\infty}\frac{Y_t^{\lambda,\rho}}{t} \geq 1-\rho-\lambda-\varepsilon\right]&.\end{aligned}$$

The first set above has probability one, because

$$\begin{aligned}&\frac{\rho(1-\rho)-\lambda(1-\lambda)}{(\rho-\lambda)} - a \geq 1-\rho-\lambda-a-\varepsilon\\ &\qquad\Leftrightarrow 1-\rho-\lambda-a \geq 1-\rho-\lambda-a-\varepsilon \Leftrightarrow 0 \geq -\varepsilon.\end{aligned}$$

We can conclude that

$$\liminf_{t\to\infty}\frac{Y_t^{\lambda,\rho}}{t} \geq 1-\rho-\lambda-\varepsilon, \qquad \text{a.s.} \tag{2.31}$$

Similarly, we can obtain the equality

$$\left[G_{at}(t)[(\sigma,\tilde{\xi}),\omega] \leq -\sum_{\substack{(a-c)t\leq x\leq at\\ x\in\mathbb{Z}}}\xi_t(x) - 1\right] = \left[Y_t^{\lambda,\rho} \leq at-ct\right], \quad \text{for some } c>0. \tag{2.32}$$

Again, using the fact that the limit (2.29) is negative if $a > 1-\lambda-\rho$, taking ε and $c>0$ such that $a-1+\lambda+\rho > \varepsilon > 0$ and $c = a-1+\lambda+\rho-\varepsilon$, we obtain that

$$\left[\frac{G_{at}(t)[(\sigma,\tilde{\xi}),\omega]}{t} \leq -\frac{1}{t}\sum_{\substack{(1-\rho-\lambda+\varepsilon)t\leq x\leq at\\ x\in\mathbb{Z}}}\xi_t(x) - \frac{1}{t}\right] = \left[\frac{Y_t^{\lambda,\rho}}{t} \leq 1-\lambda-\rho+\varepsilon\right].$$

Using the same limits as above, but in this case taking $\limsup$, it follows that

$$\begin{aligned}&[\rho(1-\rho)-\lambda(1-\lambda)-a(\rho-\lambda) \leq -(\rho-\lambda)(a-1+\lambda+\rho-\varepsilon)]\\ &= \left[\limsup_{t\to\infty}\frac{Y_t^{\lambda,\rho}}{t} \leq 1-\rho-\lambda+\varepsilon\right],\end{aligned}$$

Finally, in the same way we obtain that the first set has probability one and we can conclude that

$$\limsup_{t\to\infty} \frac{Y_t^{\lambda,\rho}}{t} \le 1-\rho-\lambda+\varepsilon, \qquad \text{a.s.} \tag{2.33}$$

The proposition follows from the arbitrariness of ε and the inequalities (2.31) and (2.33). □

Isolated second class particle

Let $\alpha \in (0,1)$. To create a isolated second class particle for the configuration η^α we consider the coupling

$$(\underset{\sim}{\eta}_t^\alpha, \tilde{\eta}_t^\alpha - \underset{\sim}{\eta}_t^\alpha), \tag{2.34}$$

where the position at time t of the second class particle in the coupling above is denoted by

$$R_t^\alpha := \{x : \tilde{\eta}_t^\alpha(x) \neq \underset{\sim}{\eta}_t^\alpha(x)\}. \tag{2.35}$$

Proposition 2.4. *We have*

$$\lim_{t\to\infty} \frac{R_t^\alpha}{t} = 1-2\alpha, \qquad a.s. \tag{2.36}$$

Proof. Take $\alpha < \rho$ and consider, as before, the coupling

$$(\underset{\sim}{\eta}_t^\alpha, \tilde{\eta}_t^\rho - \underset{\sim}{\eta}_t^\alpha) \tag{2.37}$$

and the position of the second class particle $Y_t^{\alpha,\rho}$ initially at the origin for this process. Recalling that we are using the same U and ω in the couplings (2.34) and (2.37), then we see that the second class particles in (2.37) are not in (2.34) and both R_t^α and $Y_t^{\alpha,\rho}$ have the same first class particles $\underset{\sim}{\eta}_t^\alpha$,thus, while R_t^α sees no other second class particle, $Y_t^{\alpha,\rho}$ is blocked by the second class particles $\tilde{\eta}_t^\rho - \underset{\sim}{\eta}_t^\alpha$ to its right side, this implies that

$$R_t^\alpha \ge Y_t^{\alpha,\rho}, \qquad \text{if } \alpha < \rho.$$

Hence, by the law of large number for the tagged second class particle (2.28), we obtain

$$\liminf_{t\to\infty} \frac{R_t^\alpha}{t} \ge 1-\alpha-\rho, \qquad \text{a.s.} \tag{2.38}$$

Our goal now is to prove the reverse inequality. Take now $\lambda < \alpha$ and consider the coupling

$$(\underset{\sim}{\eta}_t^\lambda, \tilde{\eta}_t^\alpha - \underset{\sim}{\eta}_t^\lambda). \tag{2.39}$$

Keep in mind that the process (2.34) has a single second class particle which starts at the origin, while the process (2.39) has many second class particles. Since $\lambda < \alpha$, we can say that all first class particles of (2.39) are present in (2.34). Therefore, the isolated second class particle of (2.34) will walk slower than the tagged second class particle of (2.39), that is,

$$R_t^\alpha \le Y_t^{\lambda,\alpha}, \qquad \text{if } \lambda < \alpha.$$

Again using the law of large number for the tagged second class particle (2.28), we obtain

$$\limsup_{t\to\infty} \frac{R_t^\alpha}{t} \le 1-\alpha-\lambda, \qquad \text{a.s.} \tag{2.40}$$

The proposition now follows from (2.38) and (2.40) taking $\rho \downarrow \alpha$ and $\lambda \uparrow \alpha$ respectively. □

2.4 Proof of hydrodynamics: increasing shock

Now, we prove the Theorem 2 for the shock case, that is $\lambda < \rho$. Recall that in this case the solution is a translation of the initial condition, i.e, $u(r,t) = u^{\lambda,\rho}(r-(1-\lambda-\rho)t)$. Let $z \in \mathbb{Z}$ and the *cut operator* define $\Gamma_z : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ as

$$\Gamma_z \eta(x) := \eta(x)\mathbb{1}_{\{x \geq z\}}. \tag{2.41}$$

This operator when applied to the configuration η eliminates the particles to the left of z.

If $\xi(0) = 1$ and Y_t is the position of the ξ particle initially at the origin, its cut the initial configuration to the left of the origin and evolve it until time t is the same as to cut the ξ_t configuration to the left of Y_t at time t. In another words,

$$(\sigma_t[(\sigma,\xi),\omega], \Gamma_{Y_t}\xi_t[(\sigma,\xi),\omega]) = (\sigma_t[(\sigma,\Gamma_0\xi),\omega], \xi_t[(\sigma,\Gamma_0\xi),\omega]). \tag{2.42}$$

This fact can be explained in words as follows. The initial ξ particles to the left to Y_0 are not felt neither by the σ particles nor by the ξ particles at Y_0 and to the right to Y_0, so it is the same to cut them at the time 0 or to cut them at time t.

Let (σ,ξ) be a two-class configuration and let the configuration η which cut the second class particles to the left of the particle at the origin (also called tagged particle, as will be seen later) and forget the class for the remaining particles. That is,

$$\eta := \sigma + \Gamma_0\xi.$$

Considering this configuration, add a second class particle with respect to η_t at the origin at time zero; calling R_t its position at time t. Add a ξ particle at the origin at time zero; call Y_t its position at time t. Then, using (2.42) we have

$$R_t = Y_t \tag{2.43}$$

$$(\underset{\sim}{\eta}_t, R_t) = (\underset{\sim}{\sigma}_t + \Gamma_{Y_t}\tilde{\xi}_t, Y_t). \tag{2.44}$$

Recall η^ρ and $\eta^{\rho,\lambda}$ are already defined as as functions of U and their diferent tilde versions are also defined in (2.13) and (2.26). Set $\sigma = \eta^\lambda$ and $\xi = \eta^\rho - \eta^\lambda$ we have of these definitions that

$$(\underset{\sim}{\sigma}, \tilde{\xi}) = (\underset{\sim}{\eta}^\lambda, \tilde{\eta}^\rho - \underset{\sim}{\eta}^\lambda)$$

$$\underset{\sim}{\eta}^{\lambda,\rho} = \underset{\sim}{\sigma} \underset{\sim}{+ \Gamma_0}\tilde{\xi}\ .$$

Let $R_t^{\lambda,\rho}$ be a position of the second class particle with respect to $\underset{\sim}{\eta}^{\lambda,\rho}$ and $Y_t^{\lambda,\rho}$ be a $\tilde{\xi}$ tagged particle for $(\underset{\sim}{\sigma}_t, \tilde{\xi}_t)$ with $R_0^{\lambda,\rho} = Y_0^{\lambda,\rho}$. Then we have for all $t \geq 0$ by (2.43) and (2.44) that

$$R_t^{\lambda,\rho} = Y_t^{\lambda,\rho} \tag{2.45}$$

$$(\underset{\sim}{\eta}_t^{\lambda,\rho}, R_t^{\lambda,\rho}) = (\underset{\sim}{\sigma}_t + \underset{\sim}{\Gamma_{Y_t^{\lambda,\rho}}}\tilde{\xi}_t, Y_t^{\lambda,\rho}). \tag{2.46}$$

Finally, by (2.46) we can write $\eta_t^{\lambda,\rho}$ as follows

$$\eta_t^{\lambda,\rho}(x) = \begin{cases} \underset{\sim}{\eta}_t^{\lambda,\rho}(x), & \text{if } x \neq R_t^{\lambda,\rho}, \\ \eta_t^{\lambda,\rho}(0), & \text{if } x = R_t^{\lambda,\rho}. \end{cases} \tag{2.47}$$

2.4.1 Proof of local equilibrium

For $\lambda < \rho$ and the solution discussed above the case in (2.11) reduces to

$$\lim_{t\to\infty} \mathbb{E}[f_A(\tau_{tr}\eta_t^{\lambda,\rho})] = \begin{cases} \rho^{|A|}, & \text{if } r > 1-\lambda-\rho, \\ \lambda^{|A|}, & \text{if } r < 1-\lambda-\rho. \end{cases} \tag{2.48}$$

We consider first the case $r > 1-\lambda-\rho$. Denote by $Y_t = Y_t^{\lambda,\rho}$ the position of the tagged ξ particle.

We clain that, if $Y_t < rt + \min A$, then

$$f_A(\tau_{tr}\eta_t^{\lambda,\rho})\mathbb{1}_{\{Y_t<rt+\min A\}} = f_A(\tau_{tr}(\sigma_t+\xi_t))\mathbb{1}_{\{Y_t<rt+\min A\}}. \tag{2.49}$$

This statement follows by the relationship (2.47) and the fact that we can drop the cut operator and the tilde operator in σ_t and ξ_t because f take values over A, whose values are above of $Y_t - rt$. Then, when we do the translation by rt, these values fall to the right of Y_t.

On the other hand,

$$\lim_{t\to\infty} \mathbb{1}_{\{Y_t\geq rt+\min A\}} = 0 \text{ a.s. and } \lim_{t\to\infty} \mathbb{1}_{\{Y_t<rt+\min A\}} = 1 \text{ a.s. }, \tag{2.50}$$

which is deduced from $\lim\limits_{t\to\infty} \dfrac{Y_t}{t} = 1-\lambda-\rho \geq r > 1-\lambda-\rho$ a.s. Therefore

$$\begin{aligned} \mathbb{E}[f_A(\tau_{tr}\eta_t^{\lambda,\rho})] &= \mathbb{E}[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\mathbb{1}_{\{Y_t<rt+\min A\}}] + \mathbb{E}[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\mathbb{1}_{\{Y_t\geq rt+\min A\}}] \\ &= \mathbb{E}[f_A(\tau_{tr}(\sigma_t+\xi_t))\mathbb{1}_{\{Y_t<rt+\min A\}}] + \mathbb{E}[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\mathbb{1}_{\{Y_t\geq rt+\min A\}}]. \end{aligned}$$

We know that $|f_A|\leq 1$ then the second summand above converges to zero using that the indicator function goes to zero a.s. and Dominated Convergence Theorem. Since $\lim\limits_{t\to\infty} \mathbb{1}_{\{Y_t<rt+\min A\}} = 1$ a.s and $\sigma_t + \xi_t = \eta_t^{\rho}$, then

$$\lim_{t\to\infty} \mathbb{E}[f_A(\tau_{tr}(\sigma_t+\xi_t))\mathbb{1}_{\{Y_t<rt+\min A\}}] = \lim_{t\to\infty} \mathbb{E}[f_A(\tau_{tr}\eta_t^{\rho})].$$

We can conclude the proof by the Lemma 1.1 since the distribution of $\tau_{tr}\eta^{\rho}$ is spatially invariant. The remaining case $r < 1-\lambda-\rho$ is similar.

2.4.2 Proof of convergence of the density fields

We use the same notation and some of the argument of the previous proof. In this part we proof (2.10), which says that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{x:at\leq x\leq bt} \eta_t^{\lambda,\rho}(x) = \int_a^b u(r,1)dr, \quad a.s.$$

Fix $a, b \in \mathbb{R}$ such that $1-\lambda-\rho < a < b$. Then

$$\frac{1}{t} \sum_{x:at\leq x\leq bt} \eta_t^{\lambda,\rho}(x) = \frac{1}{t} \sum_{x:at\leq x\leq bt} (\sigma_t(x) + \Gamma_{Y_t}\xi_t(x)) \tag{2.51}$$

$$= \frac{1}{t}\sum_{x:at\le x\le bt}(\sigma_t(x)+\xi_t(x))\mathbb{1}_{\{Y_t<at\}}+\frac{1}{t}\sum_{x:at\le x\le bt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t\ge at\}} \tag{2.52}$$

$$= \frac{1}{t}\sum_{x:at\le x\le bt}\eta_t^{\rho}(1-\mathbb{1}_{\{Y_t\ge at\}})+\frac{1}{t}\sum_{x:at\le x\le bt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t\ge at\}}$$

$$= \frac{1}{t}\sum_{x:at\le x\le bt}\eta_t^{\rho}(x)-\frac{1}{t}\sum_{x:at\le x\le bt}\eta_t^{\rho}(x)\mathbb{1}_{\{Y_t\ge at\}}+\frac{1}{t}\sum_{x:at\le x\le bt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t\ge at\}} \tag{2.53}$$

$$\underset{t\to\infty}{\longrightarrow}\rho(b-a)=\int_a^b u(r,1)\,dr,$$

where we used that: in (2.52) that we can drop the cut operator as (2.49); the limit in (2.53) follows from (2.23) for the first sum and the second and the third sums go to zero a.s. because their absolute values are bounded by $(b-a)\mathbb{1}_{\{Y_t\ge at\}}$ which goes to zero a.s. for the same reason that (2.50).

Now, consider $c, d\in\mathbb{R}$ such that $c<d<1-\lambda-\rho$. Using the same argument as before we obtain

$$\frac{1}{t}\sum_{x:ct\le x\le dt}\eta_t^{\lambda,\rho}(x)=\frac{1}{t}\sum_{x:ct\le x\le dt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))$$

$$=\frac{1}{t}\sum_{x:ct\le x\le dt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t>dt\}}+\frac{1}{t}\sum_{x:ct\le x\le dt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t\le dt\}}$$

$$=\frac{1}{t}\sum_{x:ct\le x\le dt}\eta_t^{\lambda}(x)+\frac{1}{t}\sum_{x:ct\le x\le dt}(\sigma_t(x)+\Gamma_{Y_t}\xi_t(x))\mathbb{1}_{\{Y_t\le dt\}} \tag{2.54}$$

$$\underset{t\to\infty}{\longrightarrow}\lambda(d-c)=\int_c^d u(r,1)\,dr,$$

where we obtain (2.54) since if $Y_t<dt$ and we take values between ct and dt, then we are removing the second class particles and leaving only first class particles.

Finally, we have $c, d\in\mathbb{R}$ such that $c<1-\lambda-\rho<b$ then given $\varepsilon>0$

$$\frac{1}{t}\sum_{x:ct\le x\le bt}\eta_t^{\lambda,\rho}(x)=\frac{1}{t}\sum_{x:ct\le x\le(1-\lambda-\rho-\varepsilon)t}\eta_t^{\lambda,\rho}(x)+\frac{1}{t}\sum_{x:(1-\lambda-\rho-\varepsilon)t<x<(1-\lambda-\rho+\varepsilon)t}\eta_t^{\lambda,\rho}(x)$$
$$+\frac{1}{t}\sum_{x:(1-\lambda-\rho+\varepsilon)t\le x\le bt}\eta_t^{\lambda,\rho}(x).$$

Therefore, using this equality and the above cases we can obtain that

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{x:ct\le x\le bt}\eta_t^{\lambda,\rho}(x)=\lambda(1-\lambda-\rho-\varepsilon-c)+\rho(b-1+\lambda+\rho-\varepsilon)$$
$$+\limsup_{t\to\infty}\frac{1}{t}\sum_{x:(1-\lambda-\rho-\varepsilon)t<x<(1-\lambda-\rho+\varepsilon)t}\eta_t^{\lambda,\rho}(x)$$
$$\le\lambda(1-\lambda-\rho-c)+\rho(b-1+\lambda+\rho)+\varepsilon(2-\lambda-\rho),$$

and in the same way

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{x:ct\le x\le bt} \eta_t^{\lambda,\rho}(x) \ge \lambda(1-\lambda-\rho-c) + \rho(b-1+\lambda+\rho) - \varepsilon(\lambda+\rho).$$

We can conclude, using the arbitrariness of ε, that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{x:ct\le x\le bt} \eta_t^{\lambda,\rho}(x) = \lambda(1-\lambda-\rho-c) + \rho(b-1+\lambda+\rho)$$
$$= \int_c^{1-\lambda-\rho} u(r,1)\,dr + \int_{1-\lambda-\rho}^{b} u(r,1)\,dr = \int_c^b u(r,1)\,dr.$$

2.5 Proof of hydrodynamics: rarefaction fan

Now, we prove the Theorem 2 for the rarefaction case, that is, $\lambda > \rho$. Recall that in this case the solution is given in (1.22). For this proof we need the following two results.

Lemma 2.2. *If $\lambda > \rho$ and for each $\alpha \in [0,1]$ let R_t^α be a second class particle initially at the origin for the process η_t^α. Then*

$$\eta_t^{\lambda,\rho}(x) = \begin{cases} \eta_t^\rho(x), & \text{if } x > R_t^\rho, \\ \eta_t^\lambda(x), & \text{if } x < R_t^\lambda. \end{cases} \tag{2.55}$$

Furthermore, for $\lambda \ge \alpha \ge \rho$ we have

$$\eta_t^{\lambda,\rho}(x) \le \eta_t^\alpha(x), \quad \text{for } x > R_t^\alpha, \tag{2.56}$$
$$\eta_t^\alpha(x) \le \eta_t^{\lambda,\rho}(x), \quad \text{for } x < R_t^\alpha. \tag{2.57}$$

The results in this lemma can be seen in the Figure 2.7. In the first graph we can see that $R_t^\rho \ge R_t^\lambda$ whenever $\lambda > \rho$ this follows from the fact that if $\lambda > \rho$ then $\eta_t^\lambda \ge \eta_t^\rho$, so R_t^λ moves slower that R_t^ρ because there are more first class particles interfering in a process with respect to the other.

Proof. Consider the process with only one second class particle with constant density ρ given by

$$(\underset{\sim}{\eta}_t^\rho, \tilde{\eta}_t^\rho - \underset{\sim}{\eta}_t^\rho)$$

and let R_t^ρ be the position of the second class particle for this coupling. By definition we have that $\eta^{\lambda,\rho} \ge \eta^\rho$. Then we can define the two class process whereas σ_t and ξ_t are the first and second class particles, that is,

$$(\sigma_t, \xi_t) := (\eta_t^\rho, \eta_t^{\lambda,\rho} - \eta_t^\rho).$$

Note that at time 0 $R_0^\rho = 0$ and initially the second class particles are in a negative position, i.e., $\xi(x) = \eta_t^{\lambda,\rho}(x) - \eta^\rho(x) = 0$ for all $x > 0$. The first identity in (2.55) is equivalent to

$$\xi_t(x) = 0, \quad \text{for } x > R_t^\rho. \tag{2.58}$$

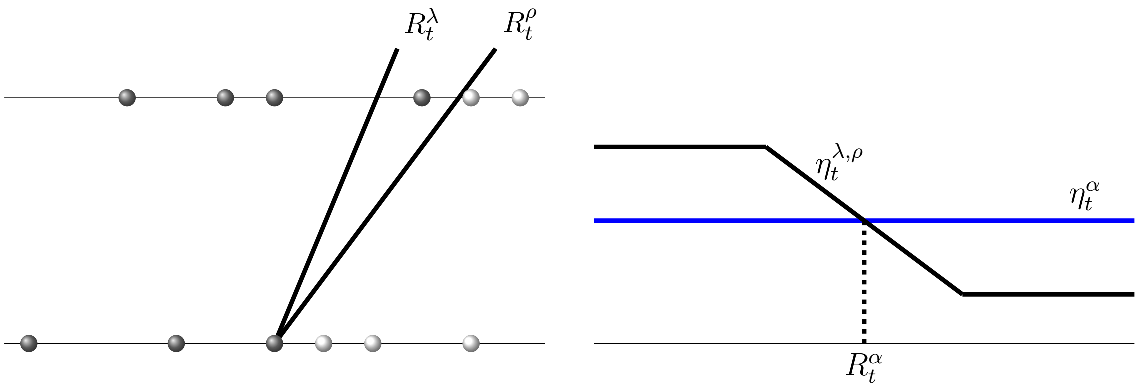

Figure 2.7. Ilustration of both results of the Lemma 2.2. In the first graph the gray and white points indicates the λ particles and ρ particles respectively.

We claim that the ξ particles cannot overpass R_t^ρ, this is

$$Y_t := \max\{y : \xi_t(y) = 1\} \leq R_t^\rho. \tag{2.59}$$

Note that Y_t behaves as a second class particle for η_t^ρ, since that the ξ particles interact by exclusion among them and we have that the rightmost ξ particle does not feel the ξ particles to its left, but with a random initial position $Y_0 \leq 0 = R_0^\rho$. To proof the claim we have to explore the following three cases, which depends of the particles in the site 0 of η^ρ and η^λ.

(a) If $\eta^\rho(0) = 0$ and $\eta^\lambda(0) = 1$, then $Y_0 = 0 = R_0^\rho$ and both particles will follow the same path in future times.

(b) If $\eta^\rho(0) = \eta^\lambda(0) = 1$, this implies that we have a first class particle at the origin in (σ_t, ξ_t) and $Y_0 < R_0^\rho$, since $\sigma_t(R_t^\rho) = 1$ and Y_t cannot jump over σ particles we obtain that $Y_t < R_t^\rho$.

(c) If $\eta^\rho(0) = \eta^\lambda(0) = 0$, it means that at the origin we have no particle and $Y_0 < R_0^\rho$. If there exist an arrow (x,t) and $Y_{t^-} = x$, $R_{t^-}^\rho = x+1$, then $Y_t = R_t^\rho = x+1$ and these continues together in higher times. Therefore, if the above is verified or not we have that $Y_t \leq R_t^\rho$.

The claim (2.59) implies (2.58), and therefore we obtain the first equality in (2.55).

To get the second identity in (2.55) we define other process of two classes. In this case the σ_t and ξ_t particles are

$$(\sigma_t, \xi_t) := (\eta_t^{\lambda,\rho}, \eta_t^\lambda - \eta_t^{\lambda,\rho}),$$

is once again well define because $\eta^\lambda \geq \eta^{\lambda,\rho}$. The second equality in (2.55) is equivalent to show that

$$\xi_t(x) = 0, \quad \text{for } x < R_t^\lambda.$$

For this argument we need a similar claim, which is:

$$Z_t := \min\{z : \xi_t(z) = 1\} \leq R_t^\lambda,$$

this follows in the same way that the above claim, following the same cases.

Finally, to prove (2.56) and (2.57) recall $\lambda \geq \alpha \geq \rho$. By definition we have that $\eta^{\lambda,\rho} \leq \eta^{\lambda,\alpha}$ and $\eta^{\alpha,\rho} \leq \eta^{\lambda,\rho}$, then by attractiveness in the Lemma 2.1 we have that $\eta_t^{\lambda,\rho} \leq \eta_t^{\lambda,\alpha}$ and $\eta_t^{\alpha,\rho} \leq \eta_t^{\lambda,\rho}$. Therefore, using the previously demonstrated results we can conclude that

$$\begin{aligned} \eta_t^{\lambda,\rho}(x) \leq \eta_t^{\lambda,\alpha}(x) = \eta_t^{\alpha}(x), &\quad \text{for } x > R_t^{\alpha}, \\ \eta_t^{\alpha}(x) = \eta_t^{\alpha,\rho}(x) \leq \eta_t^{\lambda,\rho}(x), &\quad \text{for } x < R_t^{\alpha}. \end{aligned}$$

□

Corollary 2.2. *Let $\lambda \geq \alpha > \beta \geq \rho$. Then*

$$\mathbb{P}\left(\liminf_{t\to\infty} \frac{1}{t} \sum_x \eta_t^{\lambda,\rho} \mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} \geq 2(\alpha-\beta)\beta \right) =1, \tag{2.60}$$

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{1}{t} \sum_x \eta_t^{\lambda,\rho} \mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} \leq 2(\alpha-\beta)\alpha \right) =1. \tag{2.61}$$

Proof. If $\lambda \geq \alpha > \beta \geq \rho$ this implies that $\eta_t^{\alpha} \geq \eta_t^{\beta}$ and $R_t^{\beta} \geq R_t^{\alpha}$. By (2.55) we have that $\eta_t^{\beta}(x) \leq \eta_t^{\lambda,\rho}(x)$ if $x < R_t^{\beta}$ and $\eta_t^{\alpha}(x) \geq \eta_t^{\lambda,\rho}(x)$ if $x > R_t^{\alpha}$. Therefore, putting all this together we have that

$$\sum_x \eta_t^{\beta}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},R_t^{\beta})\}} \leq \sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},R_t^{\beta})\}} \leq \sum_x \eta_t^{\alpha}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},R_t^{\beta})\}}. \tag{2.62}$$

By the law of large numbers for R_t^{α} and R_t^{β} we have that $R_t^{\alpha} \geq (1-2\alpha-\varepsilon)t$ and $R_t^{\beta} \leq (1-2\beta+\varepsilon)t$ a.s. and applying the first inequality in (2.62),

$$\begin{aligned} &\frac{1}{t}\sum_x \eta_t^{\beta}(x)\mathbb{1}_{\{x\in((1-2\alpha-\varepsilon)t,(1-2\beta+\varepsilon)t)\}} \\ &\leq \frac{1}{t}\sum_x \eta_t^{\beta}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},R_t^{\beta})\}} \\ &\leq \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},R_t^{\beta})\}} \\ &\leq \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} \\ &+\frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\alpha)t,R_t^{\alpha})\}} + \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in(R_t^{\alpha},(1-2\alpha)t)\}} \\ &+\frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in(R_t^{\beta},(1-2\beta)t)\}} + \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\beta)t),R_t^{\beta}\}} \\ &\leq \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} + \frac{2}{t}|R_t^{\alpha}-(1-2\alpha)t| + \frac{2}{t}|R_t^{\beta}-(1-2\beta)t| \\ &\leq \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} + 4\varepsilon. \end{aligned}$$

Therefore, using the law of large number (2.23) in the left side we obtain that

$$\beta(2\alpha-2\beta+2\varepsilon) \le \liminf_{t\to\infty} \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in((1-2\alpha)t,(1-2\beta)t)\}} + 4\varepsilon. \tag{2.63}$$

Therefore, by arbitrariness of ε the first part (2.60) follows. Using the same argument and the second inequality in (2.62) follows the second part (2.60) in this corollary. □

2.5.1 Proof of convergence of the density field

Fix $r \in (1-2\lambda, 1-2\rho)$. Then

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\sum_{rt<x<(1-2\rho)t}\eta_t^{\lambda,\rho}(x) &= \limsup_{t\to\infty}\frac{1}{t}\sum_{k=1}^{n}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in[t(1-2\frac{k}{n},t(1-2\frac{k-1}{n})]\cap(rt,(1-2\rho)t)\}}\\ &\le \sum_{k=1}^{n}\frac{k}{n}\frac{2}{n}\mathbb{1}_{\{\rho\le\frac{k}{n}\le\frac{1-r}{2}\}} \\ &\underset{n\to\infty}{\longrightarrow}\int_\rho^{\frac{1-r}{2}} 2r'dr' = \left(\frac{1-r}{2}\right)^2-\rho^2 = \int_r^{1-2\rho}\left(\frac{1-r'}{2}\right)dr' = \int_r^{1-2\rho} u(r',1)dr'.\end{aligned} \tag{2.64}$$

Where in (2.64) we use the second part (2.60) of previous corollary for any k with $\alpha = k/n$ and $\beta = (k-1)/n$. On the other hand,

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\sum_{rt<x<(1-2\rho)t}\eta_t^{\lambda,\rho}(x) &= \liminf_{t\to\infty}\frac{1}{t}\sum_{k=1}^{n}\sum_x \eta_t^{\lambda,\rho}(x)\mathbb{1}_{\{x\in[t(1-2\frac{k}{n},t(1-2\frac{k-1}{n})]\cap(rt,(1-2\rho)t)\}}\\ &\ge \sum_{k=1}^{n}\frac{k-1}{n}\frac{2}{n}\mathbb{1}_{\{\rho\le\frac{k}{n}\le\frac{1-r}{2}\}} \\ &\underset{n\to\infty}{\longrightarrow}\int_\rho^{\frac{1-r}{2}} 2r'dr' = \int_r^{1-2\rho} u(r',1)dr'.\end{aligned} \tag{2.65}$$

Again, we use the first part of the above corollary in (2.65) with the same α and β. By these two facts we conclude the convergence of the density fields (2.10) for intervals $(a,b)\subset(1-2\lambda,1-2\rho)$. If $a<(1-2\lambda)$ and $b>(1-2\rho)$ we claim that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{at<x<(1-2\lambda)t}\eta_t^{\lambda,\rho}(x) = \lim_{t\to\infty}\frac{1}{t}\sum_{at<x<R_t^\lambda}\eta_t^{\lambda,\rho}(x) \quad \text{a.s. and} \tag{2.66}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{(1-2\rho)t<x<bt}\eta_t^{\lambda,\rho}(x) = \lim_{t\to\infty}\frac{1}{t}\sum_{R_t^\rho<x<bt}\eta_t^{\lambda,\rho}(x) \text{ a.s..} \tag{2.67}$$

To proof the first claim note that

$$\frac{1}{t}\sum_{at<x<R_t^\lambda}\eta_t^{\lambda,\rho}(x) \le \frac{1}{t}\sum_{at<x<(1-2\lambda)t}\eta_t^{\lambda,\rho}(x) + \frac{2}{t}|R_t^\lambda-(1-2\lambda)t|.$$

Using the same inequality exchanging the sums and the law of large number of R_t^λ in (2.36) we obtain

$$\left|\frac{1}{t}\sum_{at<x<R_t^\lambda}\eta_t^{\lambda,\rho}(x)-\frac{1}{t}\sum_{at<x<(1-2\lambda)t}\eta_t^{\lambda,\rho}(x)\right|\leq\frac{2}{t}|R_t^\lambda-(1-2\lambda)t|\xrightarrow[t\to\infty]{\text{a.s.}}0.$$

This proof the first part (2.66). The second part (2.67) follows in the same way. On the other hand, using the second identity in (2.55) and the result (2.23) we obtain a.s. for t big enough

$$\left|\frac{1}{t}\sum_{at<x<R_t^\lambda}\eta_t^{\lambda,\rho}(x)-\lambda(1-2\lambda-a)\right|\leq\left|\frac{1}{t}\sum_{at<x<(1-2\lambda-\varepsilon)t}\eta_t^{\lambda}(x)-\lambda(1-2\lambda-\varepsilon-a)\right|$$
$$+\frac{1}{t}|R_t^\lambda-(1-2\lambda-\varepsilon)t|+\lambda\varepsilon\leq\varepsilon(2+\lambda).$$

This implies that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{at<x<R_t^\lambda}\eta_t^{\lambda,\rho}(x)=\lambda(1-2\lambda-a),\qquad\text{a.s. and}\tag{2.68}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{R_t^\rho<x<bt}\eta_t^{\lambda,\rho}(x)=\rho(b-(1-2\rho)),\qquad\text{a.s..}\tag{2.69}$$

Where we can obtain the equality (2.69) in the same way as (2.68). We can conclude the convergence of the density fields in the other cases using this result and the above claim, i.e.,

$$\lim_{t\to\infty}\frac{1}{t}\sum_{at<x<1-2\lambda}\eta_t^{\lambda,\rho}(x)=\lambda(1-2\lambda-a)=\int_a^{1-2\lambda}u(r',1)dr',\qquad\text{a.s. and}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{1-2\rho<x<bt}\eta_t^{\lambda,\rho}(x)=\rho(b-(1-2\rho))=\int_{1-2\rho}^{b}u(r',1)dr',\qquad\text{a.s..}$$

2.5.2 Proof of local equilibrium

Give A a finite integer set and recall $f_A(\eta)=\prod_{x\in A}\eta(x)$. Take $\lambda\geq\alpha>\beta\geq\rho$, then from (2.56) and (2.57) we have $\eta_t^\alpha(x)\geq\eta_t^{\lambda,\rho}(x)\geq\eta_t^\beta(x)$ if $x\in(R_t^\alpha,R_t^\beta)$. Hence if $y=rt+x\in(R_t^\alpha,R_t^\beta)$ with $x\in A$ then $\tau_{tr}\eta_t^\alpha(x)\geq\tau_{tr}\eta_t^{\lambda,\rho}(x)$, therefore $f_A(\tau_{tr}\eta_t^{\lambda,\rho})=0$ implies that $f_A(\tau_{tr}\eta_t^\alpha)=0$. So $f_A(\tau_{tr}\eta_t^\alpha)\geq f_A(\tau_{tr}\eta_t^{\lambda,\rho})$, and similarly we have $f_A(\tau_{tr}\eta_t^{\lambda,\rho})\geq f_A(\tau_{tr}\eta_t^\beta)$. Then, we obtain

$$B_t:=\{R_t^\alpha<x+rt<R_t^\beta,x\in A\}\subset\{f_A(\tau_{tr}\eta_t^\alpha)\geq f_A(\tau_{tr}\eta_t^{\lambda,\rho})\geq f_A(\tau_{tr}\eta_t^\beta)\}.$$

Denoting $\mathbb{1}_{B_t}$ the indicator function of the set B_t above, we have

$$\mathbb{E}\Big[f_A(\tau_{tr}\eta_t^\beta)\mathbb{1}_{B_t}\Big]\leq\mathbb{E}\Big[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\mathbb{1}_{B_t}\Big]\leq\mathbb{E}\Big[f_A(\tau_{tr}\eta_t^\alpha)\mathbb{1}_{B_t}\Big].$$

By the law of large numbers for R_t^α and R_t^β, for $r \in (1-2\alpha, 1-2\beta)$ we have that $\lim_{t\to\infty} \mathbb{1}_{B_t} = 1$ a.s. . Hence, since $|f_A| \leq 1$, for $r \in (1-2\alpha, 1-2\beta)$, using the Lemma 1.1 and the Dominated Convergence Theorem we obtain

$$\beta^{|A|} \leq \liminf_{t\to\infty} \mathbb{E}\Big[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\Big] \leq \limsup_{t\to\infty} \mathbb{E}\Big[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\Big] \leq \alpha^{|A|}.$$

Taking $\alpha \searrow \frac{1-r}{2}$ and $\beta \nearrow \frac{1-r}{2}$ we have

$$\lim_{t\to\infty} \mathbb{E}\Big[f_A(\tau_{tr}\eta_t^{\lambda,\rho})\Big] = \Big(\frac{1-r}{2}\Big)^{|A|} = u(r,1)^{|A|}.$$

This proves local equilibrium in rarefaction fan for $r \in (1-2\lambda, 1-2\rho)$. For $r \geq 1-2\rho$, by (2.55) we have $\eta_t^{\lambda,\rho}(x) = \eta_t^\rho(x)$, if $x > R_t^\rho$.Therefore, we can conclude this case using the argument in the proof above with the law of large numbers (2.36) by R_t^ρ and $B_t := \{R_t^\rho \leq x + rt, x \in A\}$. Similarly holds for $r \leq 1-2\lambda$.

Bibliography

[1] Harry Bateman. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43(4):163–170, 1915.

[2] O. Bénichou, P. Illien, G. Oshanin, A. Sarracino, and R. Voituriez. Tracer diffusion in crowded narrow channels. *Journal of Physics: Condensed Matter*, 30(44):443001, 2018.

[3] Johannes Martinus Burgers. A mathematical model illustrating the theory of turbulence. In *Advances in applied mechanics*, volume 1, pages 171–199. Elsevier, 1948.

[4] T. Chou, K. Mallick, and R. K. P. Zia. Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport. *Reports on progress in physics*, 74(11):116601, 2011.

[5] Debashish Chowdhury, Ludger Santen, and Andreas Schadschneider. Statistical physics of vehicular traffic and some related systems. *Physics Reports*, 329(4-6):199–329, 2000.

[6] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[7] Pablo A. Ferrari et al. Tasep hydrodynamics using microscopic characteristics. *Probability Surveys*, 15:1–27, 2018.

[8] Carolyn T. MacDonald and Julian H. Gibbs. Concerning the kinetics of polypeptide synthesis on polyribosomes. *Biopolymers: Original Research on Biomolecules*, 7(5):707–725, 1969.

[9] Carolyn T. MacDonald, Julian H. Gibbs, and Allen C. Pipkin. Kinetics of biopolymerization on nucleic acid templates. *Biopolymers: Original Research on Biomolecules*, 6(1):1–25, 1968.

[10] Peter V. O'Neil. *Beginning partial differential equations*, volume 85. John Wiley & Sons, 2011.

[11] Hermann Rost. Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.

[12] Timo Seppäläinen. Translation invariant exclusion processes. Available at author's website: https://people.math.wisc.edu/~seppalai/excl-book/etusivu.html, 2008.

[13] Frank Spitzer. Interaction of markov processes. In *Advance In Mathematics 5*, pages 246–290. 1970.

[14] James Stewart. *Calculo de varias variables: Conceptos y Contextos*, volume 7, pages 1128–1133. Cengage Learning, 2011.