

Hydrodynamic limit of quantum random walks

Alexandre Baraviera, Tertuliano Franco and Adriana Neumann

Abstract We discuss here the hydrodynamic limit of independent quantum random walks evolving on \mathbb{Z} . As main result, we obtain that the time evolution of the local equilibrium is governed by the convolution of the chosen initial profile with a rescaled version of the limiting probability density obtained in the law of large numbers for a single quantum random walk.

Key words: Quantum random walk, hydrodynamic limit, local equilibrium.

1 Introduction

An important subject in Statistical Physics is the comprehension of the hydrodynamic behavior of interacting particle systems. Roughly speaking, given a discrete system that evolves in time, its hydrodynamic limit consists in the limit for the time trajectory of the spatial density of particles (as some parameters are rescaled, in general, space and time). Proving rigorously such scaling limit is often a mathematical problem of deep technical difficulty. As a guide book on the subject we cite [9] and references therein.

Since the seventies, the hydrodynamic limit has been developed and successfully proved for many interacting particle systems, for instance the sym-

Alexandre Baraviera
Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil.
e-mail: baravi@mat.ufrgs.br

Tertuliano Franco
Universidade Federal da Bahia, Salvador, Brasil.
e-mail: tertu@ufba.br

Adriana Neumann
Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil.
e-mail: aneumann@impa.br

metric (and the asymmetric) simple exclusion process, the zero range process, independent random walks, and many others. In particular, the hydrodynamic limit of independent copies of a stochastic process is quite well understood, as one can see for instance in [3, 4].

We devote this paper to the study of a particular case of the hydrodynamic limit of independent copies of a stochastic process. The stochastic process we are concerned with is the *Quantum Random Walk* (QRW), as proposed in [1]. Such paper gave origin to a vast literature, inspiring several connections with quantum optics and quantum computation, see for instance the excellent survey [8] on quantum random walks.

In the recent paper [5], it was proved a law of large numbers for the QRW. Differently from the classical random walk (see [6] for a definition), the limit for the QRW, in the ballistic scale, is not a deterministic number, but a probability distribution. This is in some sense a consequence of the fact that the quantum random walk evolves faster than its classical version. In average, after the same number of steps the distance from the starting point of a quantum walk is larger than its classical counterpart.

Here, we present a proof of the hydrodynamic limit for a system of independent copies of the QRW. The hydrodynamic limit of independent copies of a stochastic process is not at all a novelty in the literature, see [4]. Nevertheless, we present these notes with the aim of introducing the QRW subject in a simple way and to make some observations on the peculiar hydrodynamic behavior for independent QRWs.

It is supposed that each QRW starts from a localized state, and the number of independent copies of the QRW starting at each state is determined by independent Poisson random variables. The parameter of each Poisson random variable is a function of space and is called the *slowly varying parameter* driven by a smooth profile γ of compact support. Under these assumptions, we prove that the limiting profile is driven by a convolution of the initial profile γ with the probability density obtained in [5] from the law of large numbers for a single quantum random walk.

It is worth to mention that, if the initial profile has compact support, then the limiting profile at any positive time will also have compact support. This contrasts with the hydrodynamic limit for classical symmetric random walks, where the limiting profile evolves according to the heat equation. For the heat equation, it is well known that the diffusion has infinite speed of propagation. That is, even for initial profiles with compact support, for any positive time, the solution will be non-zero everywhere. Hence, we roughly deduce that: while the QRW is faster than its classical counterpart (in the scaling aspect), a system of independent QRW's is slower than a system of independent classical random walks (in the macroscopic diffusion aspect).

The outline of the paper is the following: in Section 2, we define the state space of a single QRW. In Section 3, we explain the dynamics of a QRW. In Section 4 we state the hydrodynamic limit. In Section 5 we state and prove the *local equilibrium*, which in its hand implies the hydrodynamic limit.

2 The state space of the QRW

We define in this section the state space of a single QRW, which, in agreement with the postulates of the Quantum Mechanics, is a Hilbert space. Its meaning is discussed below in detail.

Definition 1. If $f_1 : H_1 \rightarrow \mathbb{C}$ and $f_2 : H_2 \rightarrow \mathbb{C}$ are two linear functionals over some vector spaces H_1 and H_2 , the tensor product of f_1 and f_2 is the bilinear functional $f_1 \otimes f_2 : H_1 \times H_2 \rightarrow \mathbb{C}$ defined by

$$(f_1 \otimes f_2)(y_1, y_2) := f_1(y_1) \cdot f_2(y_2).$$

Notice that the tensor product is bilinear, whilst its Cartesian product is linear. By the Riesz Representation Theorem, a Hilbert space can be understood as a space of linear functionals. Then, it makes sense to speak on the tensor product of two Hilbert spaces.

Definition 2. We denote the QRW state space by Ω , which is defined as the tensor product of the Hilbert spaces \mathcal{H}_P and \mathcal{H}_C :

$$\Omega := \mathcal{H}_P \otimes \mathcal{H}_C,$$

where \mathcal{H}_P is taken as the Hilbert space of square summable complex double-sided sequences:

$$\mathcal{H}_P = \ell^2(\mathbb{Z}) := \left\{ (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2 \dots) ; \sum_{k \in \mathbb{Z}} |x_k|^2 < \infty \right\},$$

being, $\forall k \in \mathbb{Z}$, $x_k \in \mathbb{C}$, and $\mathcal{H}_C = \mathbb{C}^2$.

The nomenclature \mathcal{H}_P and \mathcal{H}_C , somewhat common in the literature, comes from the idea that \mathcal{H}_P is the Hilbert space associated with the *position* of the quantum object and \mathcal{H}_C is the Hilbert space associated with the state of a certain *coin*. In the case presented here, the simplest one, \mathcal{H}_P is $\ell^2(\mathbb{Z})$ and \mathcal{H}_C is \mathbb{C}^2 .

From now on, elements of $\ell^2(\mathbb{Z})$ will be denoted by $|x\rangle$. To facilitate calculations, let $\{|e_k\rangle\}_{k \in \mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. Thus, if $x \in \ell^2(\mathbb{Z})$, then

$$|x\rangle = \sum_{k \in \mathbb{Z}} x_k |e_k\rangle.$$

According to the common notation in Quantum Mechanics, the canonical basis of \mathbb{C}^2 is denoted by $\{|+1\rangle, |-1\rangle\}$. Any element of the Hilbert space \mathbb{C}^2 is usually called a *qubit*. The qubit can be interpreted as the state of a coin (or spin).

Now, let us discuss the physical interpretation of the state space Ω . Suppose that the state of the quantum object at some time is

$$\psi := \sum_{k \in \mathbb{Z}} x_k |e_k\rangle \otimes | +1\rangle + \sum_{k \in \mathbb{Z}} y_k |e_k\rangle \otimes | -1\rangle.$$

It is a common sense in Quantum Mechanics that the particle is not at any particular place before an observation. Only after the observation, and thus after the consequently random result, one can say that the particle is at some place (more sophisticate physical interpretations are available but here we state only this pragmatic point of view).

If we perform a measurement to observe position/coin's value of the object, the outcome will be random, moreover localized, i.e. of the form $|e_k\rangle \otimes |\pm 1\rangle$, with probability proportional to the modulus' square of the respective component.

For instance, considering the state ψ above, the probability of observing the state $|e_k\rangle \otimes | +1\rangle$ as outcome (respectively $|e_k\rangle \otimes | -1\rangle$) will have probability proportional to $|x_k|^2$ (respectively $|y_k|^2$). If we perform an experiment to observe only the position, with probability proportional to $|x_k|^2 + |y_k|^2$, the outcome will be $|e_k\rangle$.

Analogously, if we perform a measurement to observe only the coin's value, the outcome will be $| +1\rangle$ with probability proportional to $\sum_{k \in \mathbb{Z}} |x_k|^2$ and the outcome will be $| -1\rangle$ with probability proportional to $\sum_{k \in \mathbb{Z}} |y_k|^2$.

3 The dynamics of a single QRW

The dynamics of the QRW is a function $U : \Omega \rightarrow \Omega$ which will be defined ahead, composed of two parts. Informally, the first part consists on an unitary operator¹ that acts on the coin. The second part is a translation to the right or to the left on the element of $\ell^2(\mathbb{Z})$, depending if the respective coin qubit is $| +1\rangle$ or $| -1\rangle$.

We recall the notation $U_2(\mathbb{C})$ for the set of unitary operators, that is, the set of linear operators on \mathbb{C}^2 preserving the canonical L^2 -norm. In this work, we treat only the particular operator $H \in U_2(\mathbb{C})$, the *Hadamard* operator, whose matrix is given in the canonical basis by

$$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (1)$$

whose effect emulates the evolution of an unbiased coin. For example, if the initial coin state is $| -1\rangle$, after the action of H we get $\frac{1}{\sqrt{2}}(| +1\rangle - | -1\rangle)$. In this final state we have one half of probability for finding one of the two possible coin states after a measurement.

We define now the part of the dynamics acting on the space. Let $\tau_m : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the shift to the right of size $m \in \mathbb{Z}$, i.e., if $|x\rangle =$

¹ Unitary matrix: its columns (or lines) compound an orthonormal basis for the space.

$\sum_{k \in \mathbb{Z}} x_k |e_k\rangle$, then

$$\tau_m |x\rangle := \sum_{k \in \mathbb{Z}} x_k |e_{k+m}\rangle.$$

The linear operator $S : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \rightarrow \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ is defined by

$$S(|x\rangle \otimes |+1\rangle) := \tau_1 |x\rangle \otimes |+1\rangle, \quad \forall x \in \ell^2(\mathbb{Z}),$$

and

$$S(|x\rangle \otimes |-1\rangle) := \tau_{-1} |x\rangle \otimes |-1\rangle, \quad \forall x \in \ell^2(\mathbb{Z}).$$

Finally, denote by Id the identity operator over $\ell^2(\mathbb{Z})$ and define $U : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \rightarrow \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by the composition

$$U := S \circ (\text{Id} \otimes H).$$

The dynamics is defined as follows: if at time zero the state is some $\psi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$, then the state at time $n = 1$ is given by $U\psi$, and at an arbitrary time $n \in \mathbb{N}$ is given by $U^n\psi$. As an example, if $\psi = |x\rangle \otimes |+1\rangle + |y\rangle \otimes |-1\rangle$, then

$$\begin{aligned} U\psi &= S(|x\rangle \otimes H(|+1\rangle) + |y\rangle \otimes H(|-1\rangle)) \\ &= S(|x\rangle \otimes \left(\frac{|+1\rangle + |-1\rangle}{\sqrt{2}}\right) + |y\rangle \otimes \left(\frac{|+1\rangle - |-1\rangle}{\sqrt{2}}\right)) \\ &= \frac{1}{\sqrt{2}} \left[(\tau_1 |x\rangle + \tau_1 |y\rangle) \otimes |+1\rangle + (\tau_{-1} |x\rangle - \tau_{-1} |y\rangle) \otimes |-1\rangle \right]. \end{aligned} \quad (2)$$

In general, for given $\psi \in \Omega = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ we will denote

$$U^n \psi = \left(\sum_{k \in \mathbb{Z}} x_{nk} |e_k\rangle \right) \otimes |+1\rangle + \left(\sum_{k \in \mathbb{Z}} y_{nk} |e_k\rangle \right) \otimes |-1\rangle, \quad (3)$$

always keeping in mind that the complex numbers x_{nk}, y_{nk} depend on ψ . Notice that the dynamics $(\psi, U\psi, U^2\psi, U^3\psi, \dots)$ is deterministic.

Definition 3. Given $\psi \in \Omega$ with unitary norm, denote by X_n^ψ a random variable (on some probability space) assuming integers values with distribution given by

$$\mathbb{P}(X_n^\psi = k) = |x_{nk}|^2 + |y_{nk}|^2, \quad \forall k \in \mathbb{Z},$$

where $x_{nk}, y_{nk} \in \mathbb{C}$ are defined in (3).

Given an initial state $\psi \in \Omega$, the state $U^n\psi$ obtained after n iterations of U gives all the information about the distribution of the position/coin of the particle at time n . Moreover, after an observation at time n of the position of the particle, the outcome of position is a random variable with the distribution presented in Definition 3.

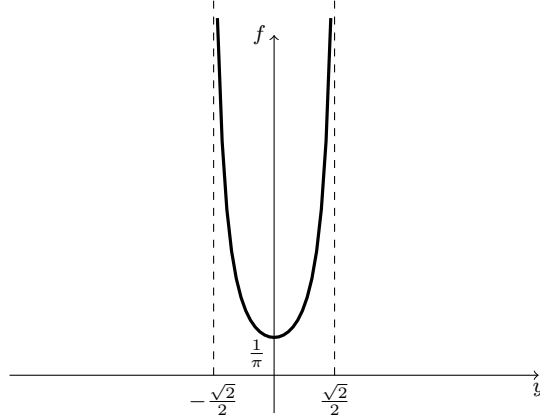


Fig. 1 Illustration of the density f obtained in the central limit theorem for the QRW, as stated in the Theorem 1.

Now we point out two remarks. First, although the Hilbert spaces are complex, if we multiply the initial state by a complex number, there is no change in the particle space distribution at final time. That is, for any $\zeta \in \mathbb{C}$ of unitary modulus, the distributions of position at time $n \in \mathbb{N}$, obtained from $U^n \psi$ and from $U^n(\zeta \psi)$ are the same. The role of complex numbers is noted in Quantum Mechanics when considering sums of states (giving rise to the phenomena known as *interference*). Second, the signs appearing in (2) generate cancellations and a very peculiar behavior² of the QRW, as one can see in the result below extracted from [5]:

Theorem 1 (Grimmett/Janson/Scudo'03). *For any $\psi \in \Omega$ which is a finite sum of localized states,*

$$\frac{X_n^\psi}{n} \xrightarrow{n \rightarrow \infty} Y, \quad \text{in distribution,}$$

where Y is a real random variable of density

$$f(y) = \begin{cases} \frac{1}{\pi(1-y^2)\sqrt{1-2y^2}}, & \text{if } y \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}], \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

² In comparison with the classical random walk.

4 Hydrodynamic limit for a system of independent quantum random walks

We turn now our attention to a system of independent QRW's. Fix once and for all a continuous non-negative function $\gamma : \mathbb{R} \rightarrow \mathbb{R}_+$ with compact support.

Definition 4. Let μ_n be a product measure on $\mathbb{N}^{\mathbb{Z}}$ whose marginal at site $k \in \mathbb{Z}$ is a $\text{Poisson}(\gamma(\frac{k}{n}))$, i.e.

$$\mu_n\{\eta \in \mathbb{N}^{\mathbb{Z}} ; \eta(k) = j\} = \frac{e^{-\lambda} \lambda^j}{j!}, \quad (5)$$

being $\lambda = \gamma(\frac{k}{n})$.

In hydrodynamic limit, this is usually called a product measure with *slowly varying parameter*, see [9]. Let $X_n^k(1), X_n^k(2), X_n^k(3), \dots$ be independent copies of the random variable X_n^ψ given in Definition 3 choosing

$$\psi = |e_k\rangle \otimes | + 1\rangle. \quad (6)$$

Remark 1. We consider initial states of the form (6) only for sake of clarity. For finite sums of localized states, all results remains in force (properly redefining the Poisson product measures above).

Denote by \mathbb{P} and \mathbb{E} the probability and the expectation, respectively, induced by μ_n and the random variables defined above. When considering a single random variable $X_n^k(j)$, we will write only P and E . Let $\mathbf{1}_A(\omega)$ be the function which is to one if $\omega \in A$ and zero otherwise.

Definition 5. For each $x \in \mathbb{Z}$, define the random variable

$$\xi_n(\ell) = \sum_{k \in \mathbb{Z}} \sum_{j=1}^{\eta(k)} \mathbf{1}_{[X_n^k(j)=\ell]}. \quad (7)$$

where $\eta \in \mathbb{N}^{\mathbb{Z}}$ is chosen according to μ_n , independently of all the random variables $X_n^k(j)$.

Intuitively, $\xi_n(x)$ is obtained by the following procedure: first we choose how many QRW's start at each localized state $|e_k\rangle \otimes | + 1\rangle$ via the measure μ_n . Then we evolve each QRW n steps. After that, we observe where each QRW is. As explained before, the outcome is random, given by some $X_n^k(j)$. Looking at (7) we notice that the random variable $\xi_n(\ell)$ counts how many of those random variables gave as result the site $\ell \in \mathbb{Z}$.

Now, we are in position to the state our main result. We denote by $C_c(\mathbb{R})$ the set of continuous functions $H : \mathbb{R} \rightarrow \mathbb{R}$ with compact support. To not overload the notation, we will write $[tn]$, the integer part of tn , only by tn .

Theorem 2 (Hydrodynamic limit of QWR's). *For all $t > 0$ and for all $H \in C_c(\mathbb{R})$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell \in \mathbb{Z}} H\left(\frac{\ell}{n}\right) \xi_{tn}(\ell) = \int_{\mathbb{R}} H(x) \rho(t, x) dx$$

in probability, where the function $\rho : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ is given by

$$\rho(t, x) = (\gamma * f_t)(x) := \int_{\mathbb{R}} \gamma(y) \frac{1}{t} f\left(\frac{x-y}{t}\right) dy, \quad (8)$$

where f is the function defined in (4) and $f_t(x) := \frac{1}{t} f\left(\frac{x}{t}\right)$.

We notice that the time $tn = \lfloor tn \rfloor$ appearing in the previous statement corresponds to the so-called *ballistic* scaling. For models where the time scaling is $\lfloor tn^2 \rfloor$, it is called the *diffusive* scaling. The limit for a system of independent quantum random walks occurs in the ballistic scaling, while for unbiased classical random walks it occurs in the diffusive scale. This is an intrinsic characteristic of quantum random walks: because of the aforementioned cancellations, they move faster than the classical random walks.

On the other hand, the time evolution of the initial profile γ according to $\gamma * f_t$ is somewhat slower than the equivalent time evolution obtained in the case of classical independent random walks, where the initial profile γ evolves through the heat equation's semigroup. For any positive time, the solution of heat equation starting from γ is positive everywhere³, but this does not happen with $\gamma * f_t$. Since f and γ have compact support, for any time $t > 0$, the function $\gamma * f_t$ has compact support as well, hence it is not positive everywhere. This means that the diffusion of mass through $\gamma * f_t$ has finite speed of propagation⁴, differently from the diffusion given by the heat equation.

5 Local equilibrium

In this section, we prove a result usually called in the literature as the *conservation of local equilibrium*, which in its hand implies the hydrodynamic limit as stated in Theorem 2.

We begin with some topological considerations. In the space $\mathbb{N}^{\mathbb{Z}}$ endowed with the distance

$$d(\eta_1, \eta_2) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \frac{|\eta_1(k) - \eta_2(k)|}{1 + |\eta_1(k) - \eta_2(k)|},$$

³ The so-called infinite propagation speed in PDE's, see [7, Page 49].

⁴ Physically, to speak about finite propagation speed for QRW's it is necessary to go further into the Lieb-Robinson bound, see [10]. We did not investigate such subject in this paper.

denote by $\{\tau_k ; k \in \mathbb{Z}\}$ the group of translations. In other words, $\tau_k \eta$ is the configuration given by

$$(\tau_k \eta)(j) = \eta(j + k).$$

The action of the translation group is naturally extended to the space of probability measures on $\mathbb{N}^{\mathbb{Z}}$. For $k \in \mathbb{Z}$ and a probability measure μ on $\mathbb{N}^{\mathbb{Z}}$, we denote by $\tau_k \mu$ the unique probability measure such that

$$\int f(\eta)(\tau_k \mu)(d\eta) = \int f(\tau_k \eta)\mu(d\eta)$$

for all integrable continuous functions f in the topology induced by the aforementioned distance.

For $c > 0$, define ν_c as the product probability measure on $\mathbb{N}^{\mathbb{Z}}$ whose marginals are Poisson probability measures with the same parameter $c > 0$, i.e.,

$$\nu_c\{\eta ; \eta(k) = \ell\} = e^{-c} \frac{c^\ell}{\ell!},$$

for any $k \in \mathbb{Z}$. Informally, the conservation of local equilibrium says that under suitable hypothesis on the initial distribution of particles, the distribution of the observed particles at time $[tn]$ is, in an asymptotic sense, locally a Poisson product measure whose parameter is a function of time and space. Its precise statement is

Theorem 3 (Conservation of local equilibrium). *Let α_{tn} be the probability measure on $\mathbb{N}^{\mathbb{Z}}$ induced by the random element*

$$\xi_{tn} := (\dots, \xi_{tn}(-2), \xi_{tn}(-1), \xi_{tn}(0), \xi_{tn}(1), \dots),$$

see the Definition 5. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, denote $xn := [xn]$. Then, for any $x \in \mathbb{R}$ and any $t > 0$,

$$\lim_{n \rightarrow \infty} \tau_{xn} \alpha_{tn} = \nu_{\rho(t,x)}$$

in the sense of weak convergence of probability measures⁵, where $\rho(t, x)$ is the function defined in (8).

Proof. In order to not overload notation, we start by considering the case $t = 1$. The general statement is postponed to the end of the proof.

The weak convergence of probability measures on $\mathbb{N}^{\mathbb{Z}}$ is equivalent to the convergence of its finite dimensional distributions, see [2]. Moreover, the convergence of the finite dimensional distributions is characterized by the convergence of their Laplace transforms. Hence, we concern our attention to the convergence of the Laplace transform

⁵ See reference [2].

$$\mathbb{E}\left[\exp\left\{-\sum_{\ell \in \mathbb{Z}} \lambda(\ell) \xi_n(\ell)\right\}\right],$$

where $\lambda : \mathbb{Z} \rightarrow \mathbb{R}_+$ is a function that is non zero only on a finite subset of \mathbb{Z} . By (7),

$$\sum_{\ell \in \mathbb{Z}} \lambda(\ell) \xi_n(\ell) = \sum_{\ell \in \mathbb{Z}} \lambda(\ell) \sum_{k \in \mathbb{Z}} \sum_{j=1}^{\eta(k)} \mathbb{1}_{[X_n^k(j)=\ell]} = \sum_{k \in \mathbb{Z}} \sum_{j=1}^{\eta(k)} \lambda(X_n^k(j)).$$

Recalling the independence of the random variables and the equality above, we obtain

$$\begin{aligned} \mathbb{E}\left[\exp\left\{-\sum_{\ell \in \mathbb{Z}} \lambda(\ell) \xi_n(\ell)\right\}\right] &= \prod_{k \in \mathbb{Z}} \mathbb{E}\left[\exp\left\{\sum_{j=1}^{\eta(k)} \lambda(X_n^k(j))\right\}\right] \\ &= \prod_{k \in \mathbb{Z}} \int E\left[\exp\left\{-\lambda(X_n^k(1))\right\}\right]^{\eta(k)} d\mu_n, \end{aligned} \quad (9)$$

where E is the expectation over a single random variable $X_n^k(1)$. Let

$$\beta_k := E\left[\exp\left\{-\lambda(X_n^k(1))\right\}\right].$$

Under μ_n , the random variable $\eta(k)$ has Poisson distribution given by (5). Thus,

$$\int \beta_k^{\eta(k)} d\mu_n = \exp\left\{\gamma\left(\frac{k}{n}\right)(\beta_k - 1)\right\}.$$

Applying this to (9) we are lead to

$$\begin{aligned} \mathbb{E}\left[\exp\left\{-\sum_{\ell \in \mathbb{Z}} \lambda(\ell) \xi_n(\ell)\right\}\right] &= \prod_{k \in \mathbb{Z}} \exp\left\{\gamma\left(\frac{k}{n}\right)(\beta_k - 1)\right\} \\ &= \exp\left\{\sum_{k \in \mathbb{Z}} \gamma\left(\frac{k}{n}\right)(\beta_k - 1)\right\}. \end{aligned} \quad (10)$$

Denote by $p_n(k, \ell)$ the probability of the quantum random walk, starting at $\psi = e_k \otimes |+1\rangle$, after a time n , have been observed at the position $\ell \in \mathbb{Z}$. That is,

$$p_n(k, \ell) := P[X_n^k(1) = \ell].$$

Therefore,

$$\beta_k := E\left[\exp\left\{-\lambda(X_n^k(1))\right\}\right] = \sum_{\ell \in \mathbb{Z}} e^{-\lambda(\ell)} p_n(k, \ell).$$

Replacing previous the formula in (10) and interchanging summations, we get to

$$\mathbb{E} \left[\exp \left\{ - \sum_{\ell \in \mathbb{Z}} \lambda(\ell) \xi_n(\ell) \right\} \right] = \exp \left\{ \sum_{\ell \in \mathbb{Z}} (e^{-\lambda(\ell)} - 1) \sum_{k \in \mathbb{Z}} \gamma\left(\frac{k}{n}\right) p_n(k, \ell) \right\}.$$

The formula above characterizes the measure α_n on $\mathbb{N}^{\mathbb{Z}}$ (induced by the random element ξ_n) as a product measure whose marginal at the site $\ell \in \mathbb{Z}$ is a Poisson probability measure of parameter

$$B(\ell, n) := \sum_{k \in \mathbb{Z}} \gamma\left(\frac{k}{n}\right) p_n(k, \ell).$$

As a consequence of symmetry of the Hadamard operator, it is easy to verify that $p_n(k, \ell) = p_n(\ell, k)$. This implies

$$B(\ell, n) = \sum_{k \in \mathbb{Z}} \gamma\left(\frac{k}{n}\right) p_n(\ell, k) = E \left[\gamma\left(\frac{X_n^\ell(1)}{n}\right) \right].$$

Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, recall the notation $xn = \lfloor xn \rfloor$. By Theorem 1, since γ is smooth, and since f is an even function,

$$\begin{aligned} \lim_{n \rightarrow \infty} B(xn, n) &= \lim_{n \rightarrow \infty} E \left[\gamma\left(\frac{X_n^{xn}}{n}\right) \right] \\ &= \lim_{n \rightarrow \infty} E \left[\gamma\left(\frac{X_n^0 + xn}{n}\right) \right] \\ &= \int_{\mathbb{R}} \gamma(y + x) f(y) dy \\ &= \int_{\mathbb{R}} \gamma(y) f(x - y) dy. \end{aligned}$$

Since α_n is a product measure, the limit above implies that

$$\lim_{n \rightarrow \infty} \tau_{xn} \alpha_n = \nu_{\rho(1, x)},$$

proving the statement for $t = 1$. For general $t > 0$, one has to replace $X_n^k(j)$ by $X_{tn}^k(j)$, keeping μ_n unchanged. Denote by α_{tn} the measure on $\mathbb{N}^{\mathbb{Z}}$ induced by ξ_{tn} . In this situation, it is straightforward to check that α_{tn} is also a product measure whose marginal has Poisson distribution and the limit for its parameter is given by

$$\begin{aligned} \lim_{n \rightarrow \infty} B(xn, tn) &= \lim_{n \rightarrow \infty} E \left[\gamma\left(\frac{X_{tn}^{xn}}{n}\right) \right] \\ &= \lim_{n \rightarrow \infty} E \left[\gamma\left(\frac{t X_{tn}^0 + xn}{tn}\right) \right] \\ &= \int_{\mathbb{R}} \gamma(ty + x) f(y) dy \\ &= \int_{\mathbb{R}} \gamma(y) \frac{1}{t} f\left(\frac{x-y}{t}\right) dy. \end{aligned}$$

Since α_{tn} is a product measure, the limit above implies that

$$\lim_{n \rightarrow \infty} \tau_{xn} \alpha_{tn} = \nu_{\rho(t,x)},$$

concluding the proof.

We shall prove Theorem 2 now.

Proof (Proof of Theorem 2). It is a known result that the conservation of local equilibrium, proved in Theorem 3, implies the hydrodynamic limit stated in Theorem 2, see for instance [9, chapter 3].

References

1. Y. Aharonov, L. Davidovich, and N. Zagury. *Quantum random walks*. Phys. Rev. A, 48(2): p.1687-1690 (1993).
2. P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics John Wiley & Sons Inc., New York, Second edition (1999).
3. A. DeMasi, E. Presutti. *Mathematical Methods for Hydrodynamic Limits*. Lecture Notes in Mathematics, Springer-Verlag, Vol. 1501 (1991).
4. R. Dobrushin, R. Siegmund-Schultze. *The Hydrodynamic Limit for Systems of Particles with Independent Evolution*. Mathematische Nachrichten, Volume 105, Issue 1, pages 199–224 (1982).
5. G. Grimmett, S. Janson, and P. Scudo. *Weak limits of quantum random walks*. Physical Review E, 69 (2004).
6. R. Durrett. *Probability: Theory and Examples*, 4th edition. Cambridge University Press, 2010.
7. L. Evans. *Partial Differential Equations*. [Graduate Studies in Mathematics v.19], American Mathematical Society, 2nd edition (1998).
8. J. Kempe, *Quantum random walks - an introductory overview*. Contemporary Physics, Vol. 44 (4), p.307-327 (2003)
9. C. Kipnis, C. Landim. *Scaling limits of interacting particle systems*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 320. Springer-Verlag, Berlin (1999).
10. E. Lieb, D. Robinson. *The Finite Group Velocity of Quantum Spin Systems*. Commun. Math. Phys. 28, 251–257 (1972).