

Non-equilibrium fluctuations for the SSEP with a slow bond

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Abstract. We prove the non-equilibrium fluctuations for the one-dimensional symmetric simple exclusion process with a slow bond. This generalizes a result of [*Stochastic Process. Appl.* **123** (2013) 4156–4185; *Stochastic Process. Appl.* **126** (2016) 3235–3242], which dealt with the equilibrium fluctuations. The foundation stone of our proof is a precise estimate on the correlations of the system, and that is by itself one of the main novelties of this paper. To obtain these estimates, we first deduce a spatially discrete PDE for the covariance function and we relate it to the local times of a random walk in a non-homogeneous environment via Duhamel's principle. Projection techniques and coupling arguments reduce the analysis to the problem of studying the local times of the classical random walk. We think that the method developed here can be applied to a variety of models, and we provide a discussion on this matter.

Résumé. Nous décrivons les fluctuations hors-équilibre du processus d'exclusion simple symétrique en dimension 1 avec des liens lents. Ceci étend un résultat de [*Stochastic Process. Appl.* **123** (2013) 4156–4185; *Stochastic Process. Appl.* **126** (2016) 3235–3242], qui traitait des fluctuations à l'équilibre. La pierre de touche de notre preuve est une estimée précise des corrélations du système, qui est en elle-même une des nouveautés principales de cet article. Pour obtenir ces estimées, nous obtenons dans un premier temps une EDP discrète en espace pour la fonction de covariance et nous la relierons aux temps locaux d'une marche aléatoire dans un environnement non-homogène, par le principe de Duhamel. Des techniques de projection et des arguments de couplage permettent de réduire l'analyse à l'étude des temps locaux de la marche aléatoire classique. Nous pensons que cette méthode peut être appliquée à une variété de modèles, et nous argumentons ce point.

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1. Introduction

One of the most challenging problems in the field of interacting particle systems is the derivation of the non-equilibrium fluctuations around the hydrodynamic limit and up to now there is not a satisfactory and robust theory that one can apply successfully. The main difficulty that one faces is to understand the precise asymptotic behaviour of the long range correlations of the system. To be more precise, when letting the interacting system start from a general measure (typically a non-invariant measure for which the hydrodynamic limit can be obtained), the correlations between any two sites are not null, but decay to zero as the scaling parameter n grows.

In many situations a uniform bound on the correlation function of order $O(1/n)$ is sufficient to obtain the non-equilibrium fluctuations of the system (see [2,17] for instance). For the model that we are going to describe in the sequel, the uniform bound on the correlation function happens to be of order $O(\log n/n)$, demanding new efforts both on the derivation of such a bound and on the application of such a bound to the proof of the non-equilibrium fluctuations.

To be more specific, here we study the symmetric simple exclusion process (SSEP) evolving on \mathbb{Z} when a slow bond is added to it. The dynamics of this model is defined as follows. On \mathbb{Z} , particles at the vertices of the bond $\{x, x + 1\}$ exchange positions at rate 1, except at the particular bond $\{0, 1\}$, where the rate of exchange is given by α/n , with $\alpha \in (0, +\infty)$. Since the rate at the bond $\{0, 1\}$ is slower with respect to the rates at other bonds, the bond $\{0, 1\}$ coined

the name *slow bond*. Particles move on the one-dimensional lattice according to those rates of exchange and they are not created nor annihilated, being the spatial disposition of particles the object of interest.

The investigation on the behaviour of this process was initiated in [3] where the hydrodynamic limit was derived (see also [5,8]). By this we mean that the density of particles of the system in the diffusive scaling converges to a function $\rho_t(\cdot)$ which is a weak solution to a partial differential equation, called the *hydrodynamic equation*. For the choice of the rates given above, the corresponding hydrodynamic equation is the one-dimensional heat equation with a boundary condition of Robin type:

$$\begin{cases} \partial_t \rho(t, u) = \partial_{uu}^2 \rho(t, u) & \text{for } u \neq 0, \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = \alpha[\rho(t, 0^+) - \rho(t, 0^-)], \\ \rho(0, u) = \rho_0(u), \end{cases} \quad (1.1)$$

where 0^+ and 0^- denote the side limits at zero from the right and from the left, respectively.

In fact, in [3] a more general choice for the rates was considered, and three different hydrodynamical behaviours were obtained. There, the slow bond was taken as the bond $\{-1, 0\}$ instead of $\{0, 1\}$, and the rate of exchange at that bond was given by $\frac{\alpha}{n^\beta}$, with $\beta \geq 0$ and α as given above. The choice of the slow bond as $\{0, 1\}$ or $\{-1, 0\}$ is essentially a matter of notation, having no special relevance. On the other hand, depending on the range of β , the boundary conditions of the hydrodynamic equation can be of Neumann type (when $\beta > 1$), which corresponds to (1.1) with $\alpha = 0$; or there is an absence of boundary conditions (when $\beta \in [0, 1)$). The model we approach here corresponds to the choice $\beta = 1$ in [3].

The effect of the slow bond at a microscopic level is obvious: it narrows down the passage of particles across it. At a macroscopic level, its presence leads to boundary conditions in the partial differential equation. By looking at the hydrodynamic equation (1.1), we see that the boundary conditions characterize the current of the system through the macroscopic position $u = 0$. The boundary conditions state that the current is proportional to the difference of concentration of the intervals $(0, +\infty)$ and $(-\infty, 0)$ near the boundary, which is in agreement with *Fick's Law*.

The *equilibrium fluctuations* for this model were presented in [4] and three different Ornstein–Uhlenbeck processes were obtained, which again had the corresponding boundary conditions as seen at the hydrodynamical level. We extend here the results of [4] by allowing the system to start from any measure and not necessarily from the stationary measure, namely the Bernoulli product measure, as required in [4]. The choice of rates as described above is restricted to $\beta = 1$ so that we are in the Robin's regime.

As the main theorem, we prove the non-equilibrium fluctuations and show that they are given by an *Ornstein–Uhlenbeck process with Robin boundary conditions*. By an Ornstein–Uhlenbeck process with Robin boundary conditions it should be understood, in the same spirit as in [4], that these boundary conditions are encoded in the space of test functions, see (2.6) below. Microscopically, the role of the boundary conditions at the level of the test functions is to force some additive functionals that appear in the Dynkin martingale to vanish as n grows. If we do not impose the boundary conditions of (2.6) on the test functions, then we would need some extra arguments to control those additive functionals. This is left to a future work.

The proof's structure is the standard one in the theory of stochastic processes: tightness for the sequence of density fields together with uniqueness of limit points. Let us discuss next the features of the work, besides the non-equilibrium result itself. And at same time we give the outline of the paper.

The biggest difficulty we face in our proof is undoubtedly the fact that the slow bond decreases the speed at which correlations vanish. In the usual SSEP, where all bonds have rate one, correlations are of order $O(1/n)$. In our case however, correlations are of order $O(\log n/n)$, therefore bigger than in the usual SSEP. For sites on the same side of the slow bond this fact is intuitive: correlations should actually increase since it is more difficult for particles to cross the slow bond. Curiously, our proof shows that the same happens for sites at different sides of the slow bond, that is, correlations are of order $O(\log n/n)$ on the entire line. An intuition of why this happens is given in Remark 4.3, and a discussion of why the bound $O(\log n/n)$ is sharp is made in Section 4.3.

In Section 2 we define the symmetric simple exclusion process in the presence of a slow bond at $\{0, 1\}$, we introduce notations and we state the main results of the article. At the end of this section, three related open problems are presented.

In Section 4 we establish connections between the two-point correlation function and the discrete derivative with the expected occupation time of a site of two-dimensional and one-dimensional random walks, respectively, in an inhomogeneous medium. This is one of the features: the way itself to estimate correlations via local times of random walks, which we believe may be applied to different contexts. The idea behind that is actually simple. We express both the discrete derivative and the correlation function as solutions to some discrete equations, then we use Duhamel's Principle to write each one of these solutions in terms of transition probabilities of random walks, in 1-d when looking at the discrete derivative and in 2-d when looking at the correlation function. Then, the local times of these random walks show up naturally from these arguments and we need to establish optimal bounds for them.

Since the necessary estimates for local times of random walks were not yet available in the literature, we derive them in Section 3 by means of projection of Markov chains (also known as *lumping*) and couplings. The statements of those estimates may look artificial at first glance, but they naturally appear when one looks for estimates on the discrete derivative of the occupation average at a site and for the two-point correlation function, as aforementioned.

As aforementioned, the (slow) rate at the edge $\{-1, 0\}$ is taken to be α/n . For the cases α/n^β , with $\beta < 1$ (as in [4]), we expect a fluctuation limit given by the usual Ornstein–Uhlenbeck as in the case $\beta = 0$ and $\alpha = 1$, in the sense of [9]. For $\beta > 1$, we expect a fluctuation limit driven by a heat equation with Neumann boundary conditions at zero, meaning that the system is split into two isolated components in the continuum limit. To adapt the present proof to these cases, it would be necessary to get the above estimates for local times of random walks. For $\beta > 1$, the adaptation of the proof seems to be straightforward, due to the fact that the estimates for $\beta = 1$ are actually deduced via estimates for a reflected random walk, which naturally pops up in the case $\beta > 1$. On the other hand, the case $\beta < 1$ seems to be more delicate: a necessary step in our proof is an estimate on the number of times a simple random walk crosses certain slow bonds. These estimates most likely break down if $\beta < 1$. Thus, some further reasoning should be necessary in this case. Here we decided to focus on the critical case $\beta = 1$ and we leave the other cases to a future work.

An additional feature is about uniqueness of the Ornstein–Uhlenbeck process with Robin boundary conditions in the non-equilibrium setting, where the variance is governed by the PDE (1.1). Suitably adapting the proofs of [9,13], we give a slightly more general version of uniqueness, which permits to consider more general starting measures than the usual slowly varying product measure. The generalization here consists on supposing that the density field associated to the initial measure does not necessarily converge to a *Gaussian field*, but only to *some field*. Moreover, this proof of uniqueness has a pedagogical importance, since the original proof of uniqueness for the Ornstein–Uhlenbeck process in the non-equilibrium setting, to the best of our knowledge, is not available in the literature.

Finally, in Section 5 we present the proof of the density fluctuations, which relies on the estimates of the discrete derivative of expected occupation number at a site, and on the two-point correlation function. A small but important detail is the fact that the estimate on the discrete derivative is sufficient for our purposes. In previous works ([2,17]), the proof of non-equilibrium fluctuations was based on the convergence of the spatially discretized heat equation towards the continuum heat equation. Such an approximation is quite good, of order $O(n^{-2})$, and quite hard to adapt to the non-homogeneous medium set up without some uniform ellipticity assumption as in [12]. On the other hand, the discrete derivative estimate for the spatially discretized PDE is much easier to reach, as seen here. This idea on making use of the discrete derivative first appeared in [7], but its utility becomes more evident now.

2. Statement of results

2.1. The model

We fix a parameter $\alpha > 0$, and we consider the symmetric simple exclusion process $\{\eta_t : t \geq 0\}$ with a slow bond as defined in [3]. More precisely, $\{\eta_t : t \geq 0\}$ is the Markov process with state space $\Omega \stackrel{\text{def}}{=} \{0, 1\}^{\mathbb{Z}}$, and infinitesimal generator \mathcal{L}_n acting on local functions $f : \Omega \rightarrow \mathbb{R}$ via

$$(\mathcal{L}_n f)(\eta) = \sum_{x \in \mathbb{Z}} \xi_{x,x+1}^n (f(\eta^{x,x+1}) - f(\eta)), \tag{2.1}$$

where

$$\xi_{x,x+1}^n \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \neq 0, \\ \frac{\alpha}{n} & \text{if } x = 0. \end{cases} \tag{2.2}$$

Here, for any $x \in \mathbb{Z}$, the configuration $\eta^{x,x+1}$ is obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x + 1)$, i.e.,

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x + 1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x + 1, \\ \eta(y) & \text{otherwise,} \end{cases}$$

see Figure 1 for an illustration of the jump rates. Given $\eta \in \{0, 1\}^{\mathbb{Z}}$, we then say that the site $x \in \mathbb{Z}$ is vacant if $\eta(x) = 0$ and occupied if $\eta(x) = 1$.

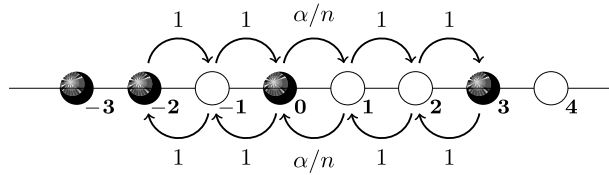


Fig. 1. Jump rates. The bond $\{0, 1\}$ has a particular jump rate associated to it, which is given by α/n .

2.2. Hydrodynamic limit

Fix a measurable density profile $\rho_0 : \mathbb{R} \rightarrow [0, 1]$. For each $n \in \mathbb{N}$, let μ_n be a probability measure on Ω . We say that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is associated with the profile $\rho_0(\cdot)$ if, for any $\delta > 0$ and any continuous function of compact support $f : \mathbb{R} \rightarrow \mathbb{R}$, the following holds:

$$\lim_{n \rightarrow \infty} \mu_n \left[\eta : \left| \frac{1}{n} \sum_{x \in \mathbb{Z}} f\left(\frac{x}{n}\right) \eta(x) - \int f(u) \rho_0(u) du \right| > \delta \right] = 0. \tag{2.3}$$

Fix $T > 0$, and let $\mathcal{D}([0, T], \Omega)$ be the space of trajectories which are right continuous, with left limits and taking values in Ω . Denote by \mathbb{P}_{μ_n} the probability measure on $\mathcal{D}([0, T], \Omega)$ induced by the SSEP with a slow bond accelerated by n^2 , i.e., the Markov process with generator $n^2 \mathcal{L}_n$, and initial measure μ_n . With a slight abuse of notation, we also use the notation $\{\eta_t : t \in [0, T]\}$ for the accelerated process. Denote by \mathbb{E}_{μ_n} the expectation with respect to \mathbb{P}_{μ_n} . In [3,5] the hydrodynamical behaviour was studied. We note that the process there was studied in finite volume, i.e., the model was considered on the discrete torus embedded into the continuous one-dimensional torus. However, since the extension to infinite volume is just a topological issue, the statement below can be obtained via an adaptation of the original approach:

Theorem 2.1 ([3,5]). *Suppose that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is associated to the profile $\rho_0(\cdot)$ in the sense of (2.3). Then, for each $t \in [0, T]$, for any $\delta > 0$ and any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support,*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_{\mu_n} \left[\eta : \left| \frac{1}{n} \sum_{x \in \mathbb{Z}} f\left(\frac{x}{n}\right) \eta_t(x) - \int_{\mathbb{R}} f(u) \rho(t, u) du \right| > \delta \right] = 0,$$

where $\rho(t, \cdot)$ is the unique weak solution of the heat equation with Robin boundary conditions given by

$$\begin{cases} \partial_t \rho(t, u) = \partial_{uu}^2 \rho(t, u), & t \geq 0, u \in \mathbb{R} \setminus \{0\}, \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = \alpha[\rho(t, 0^+) - \rho(t, 0^-)], & t \geq 0, \\ \rho(0, u) = \rho_0(u), & u \in \mathbb{R}. \end{cases} \tag{2.4}$$

Here, $\rho(t, 0^+)$ and $\rho(t, 0^-)$ denote the limit from the right and from the left at zero, respectively. The notation 0^\pm will be used throughout the article.

2.3. Space of test functions and semigroup

In this section we introduce a space of test functions, that is suitable for our purposes, and which, basically, coincides with the one in [6]. Here, functions are continuous from the left at zero, while in [6] functions are continuous from the right. This subtle difference is due to the choice of slow bond's position, which is $\{0, 1\}$ here and $\{-1, 0\}$ in [6].

Definition 1. We denote by $\mathcal{S}_\alpha(\mathbb{R})$ the space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that:

- (i) f is smooth on $\mathbb{R} \setminus \{0\}$, i.e. $f \in C^\infty(\mathbb{R} \setminus \{0\})$,
- (ii) f is continuous from the left at 0,
- (iii) for all non-negative integers k, ℓ , the function f satisfies

$$\|f\|_{k, \ell} := \sup_{u \neq 0} \left| (1 + |u|^\ell) \frac{d^k f}{du^k}(u) \right| < \infty. \tag{2.5}$$

(iv) for any integer $k \geq 0$,

$$\frac{d^{2k+1}f}{du^{2k+1}}(0^+) = \frac{d^{2k+1}f}{du^{2k+1}}(0^-) = \alpha \left(\frac{d^{2k}f}{du^{2k}}(0^+) - \frac{d^{2k}f}{du^{2k}}(0^-) \right). \quad (2.6)$$

Moreover, $\mathcal{S}'_\alpha(\mathbb{R})$ denotes the topological dual of $\mathcal{S}_\alpha(\mathbb{R})$.

In plain words, $\mathcal{S}_\alpha(\mathbb{R})$ essentially consists of the space of functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ that are not necessarily smooth at the origin. It is a consequence of (2.5) that $\frac{d^k f}{du^k}(0^+)$ and $\frac{d^k f}{du^k}(0^-)$ exist for all integers $k \geq 0$. As in [4], one may show that $\mathcal{S}_\alpha(\mathbb{R})$ is a Fréchet space (this fact was only used when showing tightness, see [16]). We recall below the explicit formula for the semigroup that corresponds to the PDE (2.4).

Proposition 2.2 ([4]). *Denote by g_{even} and g_{odd} the even and odd parts of a function $g : \mathbb{R} \rightarrow \mathbb{R}$, respectively. That is, for $u \in \mathbb{R}$,*

$$g_{\text{even}}(u) = \frac{g(u) + g(-u)}{2} \quad \text{and} \quad g_{\text{odd}}(u) = \frac{g(u) - g(-u)}{2}.$$

The solution of (2.4) with initial condition $g \in \mathcal{S}_\alpha(\mathbb{R})$ is given by

$$T_t^\alpha g(u) = \frac{1}{\sqrt{4\pi t}} \left\{ \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{4t}} g_{\text{even}}(y) dy + e^{2\alpha u} \int_u^{+\infty} e^{-2\alpha z} \int_0^{+\infty} \left[\left(\frac{z-y+4\alpha t}{2t} \right) e^{-\frac{(z-y)^2}{4t}} + \left(\frac{z+y-4\alpha t}{2t} \right) e^{-\frac{(z+y)^2}{4t}} \right] g_{\text{odd}}(y) dy dz \right\},$$

for $u > 0$, and

$$T_t^\alpha g(u) = \frac{1}{\sqrt{4\pi t}} \left\{ \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{4t}} g_{\text{even}}(y) dy - e^{-2\alpha u} \int_{-u}^{+\infty} e^{-2\alpha z} \int_0^{+\infty} \left[\left(\frac{z-y+4\alpha t}{2t} \right) e^{-\frac{(z-y)^2}{4t}} + \left(\frac{z+y-4\alpha t}{2t} \right) e^{-\frac{(z+y)^2}{4t}} \right] g_{\text{odd}}(y) dy dz \right\},$$

for $u < 0$.

The next proposition connects T_t^α with the space of test functions $\mathcal{S}_\alpha(\mathbb{R})$.

Proposition 2.3 ([6]). *The operator T_t^α defines a semigroup $T_t^\alpha : \mathcal{S}_\alpha(\mathbb{R}) \rightarrow \mathcal{S}_\alpha(\mathbb{R})$. That is, for any given $g \in \mathcal{S}_\alpha(\mathbb{R})$ and any time $t > 0$, the solution $T_t^\alpha g$ of the PDE (2.4) starting from g also belongs to $\mathcal{S}_\alpha(\mathbb{R})$.*

Definition 2. Let $\Delta_\alpha : \mathcal{S}_\alpha(\mathbb{R}) \rightarrow \mathcal{S}_\alpha(\mathbb{R})$ be the Laplacian on $\mathcal{S}_\alpha(\mathbb{R})$, i.e., for any $f \in \mathcal{S}_\alpha(\mathbb{R})$,

$$\Delta_\alpha f(u) = \begin{cases} \partial_{uu}^2 f(u) & \text{if } u \neq 0, \\ \partial_{uu}^2 f(0^+) & \text{if } u = 0. \end{cases} \quad (2.7)$$

The definition of the operator $\nabla_\alpha : \mathcal{S}_\alpha(\mathbb{R}) \rightarrow C^\infty[0, 1]$ is analogous.

2.4. Discrete derivatives and covariance estimatives

Fix an initial measure μ_n on Ω . For $x \in \mathbb{Z}$ and $t \geq 0$, let

$$\rho_t^n(x) \stackrel{\text{def}}{=} \mathbb{E}_{\mu_n}[\eta_t(x)]. \quad (2.8)$$

A simple computation shows that $\rho_t^n(\cdot)$ is a solution of the discrete equation

$$\partial_t \rho_t^n(x) = (n^2 \mathcal{A}_n \rho_t^n)(x), \quad x \in \mathbb{Z}, t \geq 0, \quad (2.9)$$

where the operator \mathcal{A}_n acts on functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ as

$$(\mathcal{A}_n f)(x) := \xi_{x,x+1}^n (f(x+1) - f(x)) + \xi_{x-1,x}^n (f(x-1) - f(x)), \quad \forall x \in \mathbb{Z}, \tag{2.10}$$

with $\xi_{x,x+1}^n$ as defined in (2.2).

Definition 3. For $x, y \in \mathbb{Z}$, and $t \in [0, T]$, define the two-point correlation function

$$\varphi_t^n(x, y) \stackrel{\text{def}}{=} \mathbb{E}_{\mu_n} [\eta_t(x)\eta_t(y)] - \rho_t^n(x)\rho_t^n(y). \tag{2.11}$$

We now state two results that are fundamental for the study of density fluctuations, which are interesting by themselves.

Theorem 2.4 (Discrete derivative estimate). Assume that there exists a constant $c > 0$ that does not depend on n such that

$$\sup_{x \in \mathbb{Z}} \left| \rho_0^n(x) - \rho_0 \left(\frac{x}{n} \right) \right| \leq \frac{c}{n}. \tag{2.12}$$

Then, there exists $\bar{c} = \bar{c}(T, \alpha, \rho_0) > 0$ such that, for all $t \in [0, T]$, and all $n \in \mathbb{N}$,

$$|\rho_t^n(x+1) - \rho_t^n(x)| \leq \begin{cases} \frac{\bar{c}}{n} & \text{if } x \neq 0, \\ \bar{c} & \text{if } x = 0. \end{cases}$$

Note that the second inequality above is obvious, but we kept in the statement of the theorem for the sake of clarity.

Theorem 2.5 (Correlation estimate). Let $V := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \geq x + 1\}$. Assume that there exists a constant $c > 0$ that does not depend on n such that

$$\sup_{(x,y) \in V} |\varphi_0^n(x, y)| \leq \frac{c}{n}. \tag{2.13}$$

Moreover, assume that (2.12) is satisfied. Then, there exists $\hat{c} = \hat{c}(T, \alpha, \rho_0) > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{t \leq T} \sup_{(x,y) \in V} |\varphi_t^n(x, y)| \leq \frac{\hat{c} \log n}{n}. \tag{2.14}$$

Remark 2.6. Note that by the symmetry of the correlation function, Theorem 2.5 immediately implies (2.14) for $x \neq y$. We state (2.14) in this form to highlight the set V , which plays a special role in the proof of (2.14).

2.5. Ornstein–Uhlenbeck process

Let $\rho(t, \cdot)$ be the unique solution of the hydrodynamic equation (2.4). In what follows, $\mathcal{D}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ (resp. $\mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$) denotes the space of càdlàg (resp. continuous) $\mathcal{S}'_\alpha(\mathbb{R})$ valued functions endowed with the Skorohod topology. We also denote by χ the static compressibility defined by $\chi(\rho) = \rho(1 - \rho)$. Denote by $\langle \cdot, \cdot \rangle_{\rho_t(\cdot)}$ the inner product with respect to $L^2_{\Lambda_t}(\mathbb{R})$, where the measure $\Lambda_t(du)$ is given by

$$\Lambda_t(du) \stackrel{\text{def}}{=} 2\chi(\rho_t(u)) du + \frac{1}{\alpha} [\rho_t(0^-)(1 - \rho_t(0^+)) + \rho_t(0^+)(1 - \rho_t(0^-))] \delta_0(du), \tag{2.15}$$

where $\delta_0(du)$ denotes the Dirac measure at zero. More precisely, for $f, g \in \mathcal{S}_\alpha(\mathbb{R})$,

$$\langle f, g \rangle_{\rho_t(\cdot)} = \int_{\mathbb{R}} 2\chi(\rho_t(u)) f(u)g(u) du + \frac{1}{\alpha} [\rho_t(0^-)(1 - \rho_t(0^+)) + \rho_t(0^+)(1 - \rho_t(0^-))] f(0)g(0).$$

Proposition 2.7. There exists a unique (in distribution) random element \mathcal{Y} taking values in the space $\mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ such that the following two conditions hold:

(i) For every function $f \in \mathcal{S}_\alpha(\mathbb{R})$, the stochastic processes $\mathcal{M}_t(f)$ and $\mathcal{N}_t(f)$ given by

$$\mathcal{M}_t(f) = \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s(\Delta_\alpha f) ds, \tag{2.16}$$

$$\mathcal{N}_t(f) = (\mathcal{M}_t(f))^2 - \int_0^t \|\nabla_\alpha f\|_{\rho_s(\cdot)}^2 ds \tag{2.17}$$

are \mathcal{F}_t -martingales, where for each $t \in [0, T]$, $\mathcal{F}_t := \sigma(\mathcal{Y}_s(f); s \leq t, f \in \mathcal{S}_\alpha(\mathbb{R}))$.

(ii) \mathcal{Y}_0 is a random element taking values in $\mathcal{S}'_\alpha(\mathbb{R})$ with a fixed distribution.

Moreover, if (i) and (ii) hold, then:

- for each $f \in \mathcal{S}_\alpha(\mathbb{R})$, conditionally to \mathcal{F}_s with $s < t$, the distribution of $\mathcal{Y}_t(f)$ is normal of mean $\mathcal{Y}_s(T_{t-s}^\alpha f)$ and variance $\int_s^t \|\nabla_\alpha T_r^\alpha f\|_{\rho_r(\cdot)}^2 dr$.
- If \mathcal{Y}_0 is a Gaussian field, then the stochastic process $\{\mathcal{Y}_t(f); t \geq 0\}$ will be Gaussian indeed.

In other words, if \mathcal{Y}^1 and \mathcal{Y}^2 are two random elements taking values on $\mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ and satisfying the martingale problem described above by (i) and (ii), then \mathcal{Y}^1 and \mathcal{Y}^2 must have the same distribution.

It is common in the literature to write the martingale problem stated above as a *formal* solution of some generalized stochastic partial differential equation. We discuss it with no mathematical rigour, aiming only at giving some intuition on the fluctuations' global behaviour.

We call the random element \mathcal{Y} defined via Proposition 2.7 a generalized Ornstein–Uhlenbeck process which takes values in $\mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ and it is the *formal* solution of

$$d\mathcal{Y}_t = \Delta_\alpha \mathcal{Y}_t dt + \nabla_\alpha d\mathcal{W}_t, \tag{2.18}$$

where:

- The operators Δ_α and ∇_α have been given in Definition 2 and are usually referred to as the *characteristics* of the Ornstein–Uhlenbeck process.
- \mathcal{W} is a space–time white noise with respect to the measure $\Lambda_s(du)$, i.e., \mathcal{W} is a mean-zero Gaussian random element taking values in the dual space of $L^2_\Lambda([0, \infty) \times \mathbb{R})$ with covariances given by

$$\mathbb{E}[\mathcal{W}(F)\mathcal{W}(G)] = \int_0^\infty \int_{\mathbb{R}} F(s, u)G(s, u) d\Lambda(s, u), \quad \forall F, G \in L^2_\Lambda([0, \infty) \times \mathbb{R}),$$

where $d\Lambda(s, u) = d\Lambda_s(u) \times ds$, and Λ_s has been defined in (2.15).

- For $f \in \mathcal{S}_\alpha(\mathbb{R})$, we define $\mathcal{W}_t(f) := \mathcal{W}(f\mathbf{1}_{[0,t]})$. In particular, $\{\mathcal{W}_t(f) : f \in \mathcal{S}_\alpha(\mathbb{R})\}$ is a Gaussian process with covariance given on $f, g \in \mathcal{S}_\alpha(\mathbb{R})$ by

$$\mathbb{E}[\mathcal{W}_t(f)\mathcal{W}_t(g)] = \int_0^t \langle f, g \rangle_{\rho_s(\cdot)} ds.$$

2.6. Non-equilibrium fluctuations

We define the density fluctuation field \mathcal{Y}^n as the time-trajectory of a linear functional acting on functions $f \in \mathcal{S}_\alpha(\mathbb{R})$ via

$$\mathcal{Y}_t^n(f) \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} f\left(\frac{x}{n}\right) (\eta_t(x) - \rho_t^n(x)). \tag{2.19}$$

For each $n \geq 1$, let \mathcal{Q}_n be the probability measure on $\mathcal{D}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ induced by the density fluctuation field \mathcal{Y}^n and a measure μ_n . We now state the main result of this paper:

Theorem 2.8 (Non-equilibrium fluctuations). *Consider the Markov processes $\{\eta_t : t \geq 0\}$ starting from a sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ associated with a profile as in (2.3), and assume:*

- (A) Conditions (2.12) and (2.13) on mean and covariance, respectively.
- (B) There exists a $\mathcal{S}'_\alpha(\mathbb{R})$ -valued random variable \mathcal{Y}_0 such that \mathcal{Y}_0^n converges in distribution to \mathcal{Y}_0 , whose law we denote by \mathcal{L} .

Then, the sequence of processes $\{\mathcal{Y}_t^n\}_{n \in \mathbb{N}}$ converges in distribution, as $n \rightarrow +\infty$, with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$ to a random element \mathcal{Y} in $\mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$, the generalized Ornstein–Uhlenbeck which is a solution of (2.18), and \mathcal{Y}_0 has law \mathcal{L} .

It is of worth to give examples of sequences $\{\mu_n\}_{n \in \mathbb{N}}$ of initial measures satisfying assumptions (A) and (B). Next, we present two examples of such initial measures and we leave an open question on the subject.

The first example we present is the standard one for non-equilibrium fluctuations: take $\{\mu_n\}_{n \in \mathbb{N}}$ as the slowly varying Bernoulli product measure $\{\nu_{\rho_0(\cdot)}^n\}_{n \in \mathbb{N}}$ associated with a smooth profile $\rho_0 : \mathbb{R} \rightarrow [0, 1]$, that is, $\nu_{\rho_0(\cdot)}^n$ is a product measure on $\{0, 1\}^{\mathbb{Z}}$ such that

$$\nu_{\rho_0(\cdot)}^n \left\{ \eta \in \{0, 1\}^{\mathbb{Z}} : \eta(x) = 1 \right\} = \rho_0 \left(\frac{x}{n} \right).$$

Obviously, (A) is satisfied. The proof that (B) holds is just an adaptation of the analogous result for the SSEP, being included in Proposition B.1 for the sake of completeness.

The second example we discuss is somewhat artificial, but, in any case, illustrates the existence of a sequence of non-product measures satisfying (A) and (B). Let μ_n be the measure on Ω induced by the distribution at the time rn^2 , where $r > 0$ is fixed, of the (homogeneous) one-dimensional SSEP started from the slowly varying measure $\nu_{\rho_0(\cdot)}^n$ defined above.

From the *propagation of local equilibrium* for the SSEP (see [13] and references therein), one can check that condition (2.12) holds. Besides that, it is well known that the SSEP has long range correlations of order $O(1/n)$, giving (2.13). Thus, assumption (A) is satisfied. From the non-equilibrium fluctuations for the homogeneous SSEP (see [2,17]) one can deduce that (B) is satisfied, where the law \mathcal{L} is determined by the distribution of the Ornstein–Uhlenbeck process at time $r > 0$.

We now debate the issue of which properties a sequence of initial measures should have in order to satisfy (A) and (B). Assume, for the moment, that the initial measures $\{\mu_n\}_{n \in \mathbb{N}}$ for the Markov processes $\{\eta_t : t \geq 0\}$ satisfy:

- (i) Condition (2.12) holds.
- (ii) For each $n \in \mathbb{N}$, the correlation at the initial time is of order $O(1/n)$ times a bounded profile ζ^n , that is,

$$\varphi_0^n(x, y) = \frac{\zeta^n \left(\frac{x}{n}, \frac{y}{n} \right)}{n}, \quad \forall x, y \in \mathbb{Z}, \forall n \in \mathbb{N},$$

where the sequence of functions $\zeta^n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ converges uniformly to a bounded continuous function $\zeta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ as $n \rightarrow \infty$. Note that this implies (2.13).

Under (i) and (ii), condition (A) holds. Moreover, under (i) and (ii), and following the same steps of Section 5.2, one can obtain tightness of $\{\mathcal{Y}_0^n\}_{n \in \mathbb{N}}$. Hence, in order to achieve (B), it is only missing the convergence in distribution of the sequence of initial density fields $\{\mathcal{Y}_0^n\}_{n \in \mathbb{N}}$. Let $f, g \in \mathcal{S}_\alpha(\mathbb{R})$. By simple calculations,

$$\mathbb{E}_{\mu_n} [\mathcal{Y}_0^n(f) \mathcal{Y}_0^n(g)] = \frac{1}{n} \sum_{x \in \mathbb{Z}} f \left(\frac{x}{n} \right) g \left(\frac{x}{n} \right) \mathbb{E}_{\mu_n} [(\bar{\eta}_0(x))^2] + \frac{1}{n} \sum_{\substack{x \neq y \\ x, y \in \mathbb{Z}}} f \left(\frac{x}{n} \right) g \left(\frac{y}{n} \right) \varphi_0^n(x, y).$$

Above $\bar{\eta}$ denotes the centered random variable $\eta : \bar{\eta}_t(x) := \eta_t(x) - \rho_t^n(x)$. Under (i) and (ii), it is easy to check that expression above converges to

$$\int_{\mathbb{R}} \chi(\rho_0(u)) f(u) g(u) du + \int_{\mathbb{R}} \int_{\mathbb{R}} \zeta(w, r) f(w) g(r) dw dr$$

as $n \rightarrow \infty$. Note that the limit above indicates that $1/n$ is the right order on the decay of correlations in order to exist a limiting non zero effect on the distribution of initial density field \mathcal{Y}_0 . However, convergence of means and decay of correlations do not suffice to assure that $\mathcal{Y}_0^n(f)$ actually converges in distribution: some special central limit theorem is required here. This CLT is not an easy subject due to the slow decay of correlations and due to the fact that for each n , the random variables $\eta_0(x)$ may have different distributions. We therefore leave it as an open question:

Open Question 1. *Given assumption (A) of Theorem 2.8, which additional hypotheses are necessary for (B) to hold?*

Without going into details, we affirm that a natural strategy to prove current/tagged particle fluctuations relies on a decay of correlations of order $O(1/n)$, see [11]. However, the correlations of the non-equilibrium SSEP with a slow

bond here considered are of order $O(\log n/n)$, see Theorem 2.5. Moreover, the current/tagged particle fluctuations for the equilibrium scenario with a slow bond are already understood, see [4]. This leads us to:

Open Question 2. *How to prove current/tagged particle fluctuations for the non-equilibrium SSEP with a slow bond? May (or must) a different scaling be considered?*

Finally, naturally inspired by [4], we state:

Open Question 3. *Consider $\beta > 0$ with $\beta \neq 1$. How to prove non-equilibrium fluctuations for the one-dimensional SSEP with a slow bond of rate $\alpha n^{-\beta}$?*

We believe that this last open problem shall be solved by the methods of this paper, and we leave it for a future work. For the first two problems, we have no clear strategy to solve them.

3. Estimates on local times

In this section we derive estimates on the local times of a random walk with inhomogeneous rates, which will be later used in Section 4 in the proofs of Theorems 2.4 and 2.5. In the sequel, given any Markov chain Z and a set A , we denote by $L_t(A)$ the local time of Z in A until time t :

$$L_t(A) \stackrel{\text{def}}{=} \int_0^t \mathbf{1}_{\{Z_s \in A\}} ds. \tag{3.1}$$

3.1. Estimates in dimension two

We denote by $\{(\mathbf{X}_t, \mathbf{Y}_t); t \geq 0\}$ the random walk on the set $V = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \geq x + 1\}$ with generator \mathbf{B}_n acting on local functions $f : V \rightarrow \mathbb{R}$ via

$$(\mathbf{B}_n f)(u) \stackrel{\text{def}}{=} \sum_{v \in V} c_n(u, v) [f(v) - f(u)], \quad \forall u \in V. \tag{3.2}$$

Here, the rates are defined as pictured in Figure 2. More precisely, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that the L^1 -norm¹ satisfies $\|u - v\|_1 = 1$, we define

$$c_n(u, v) \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha}{n} & \text{if } (u, v) \in \mathcal{U}, \\ 1 & \text{if } u \notin \mathcal{U} \text{ or } v \notin \mathcal{U} \end{cases}$$

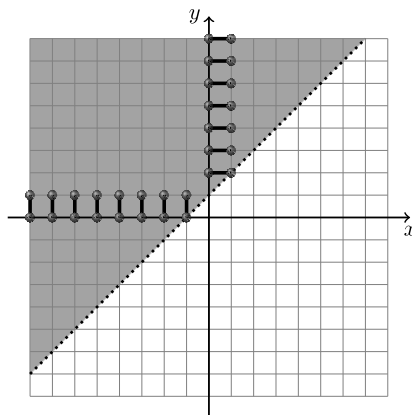


Fig. 2. Sets V , D and U and \mathcal{U} . Sites of V are the ones laying on the light gray region. Sites in D lay on the dotted line and sites of U are marked as gray balls. Elements of \mathcal{U} are edges marked with a thick black segment having jump rate equal to α/n (slow bonds). Any other edges have rate 1.

¹We write $\|\cdot\|_1$ for the L^1 -norm on \mathbb{Z}^2 , that is, $\|(u_1, u_2)\|_1 = |u_1| + |u_2|$.

and $c_n(u, v) = 0$ if the L^1 -distance of u and v is not equal to one. Here, U is the subset of V given by

$$U \stackrel{\text{def}}{=} \{(x, y) \in V : x \in \{0, 1\} \text{ and } y \geq 2\} \cup \{(x, y) \in V : x \leq -1 \text{ and } y \in \{0, 1\}\},$$

and \mathcal{U} is the subset of $U^{\otimes 2}$ defined via

$$\begin{aligned} \mathcal{U} \stackrel{\text{def}}{=} & \{(u, v) \in U^{\otimes 2} : \|u - v\| = 1, |u_1 - v_1| = 1\} \\ & \cup \{(u, v) \in U^{\otimes 2} : \|u - v\| = 1, |u_2 - v_2| = 1\}. \end{aligned} \tag{3.3}$$

We furthermore denote by D the ‘‘upper diagonal’’ defined by $D \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : y = x + 1\}$, see Figure 2.

By $\mathbf{E}_{(x,y)}$, and $\mathbf{P}_{(x,y)}$ we denote the corresponding probability and expectation when starting from $(x, y) \in V$. The goal of this section is to prove the following result.

Proposition 3.1. *There exists a constant $c > 0$ such that for all $(x, y) \in V$, all $n \in \mathbb{N}$, and all $t \geq 0$,*

$$\begin{aligned} \mathbf{E}_{(x,y)}[L_{tn^2}(D \setminus \{(0, 1)\})] &\leq cn\sqrt{t}, \quad \text{and} \\ \mathbf{E}_{(x,y)}[L_{tn^2}(\{(0, 1)\})] &\leq c \log(tn^2). \end{aligned} \tag{3.4}$$

To prove Proposition 3.1, we estimate first in Lemma 3.2 the local time of a simple random walk confined to the boundary of the set

$$W \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : 0 \leq x \leq y\},$$

which is V intersected with the first quadrant shifted by the vector $(1, 2)$. In plain words we identified the vertex $(1, 2)$ in V with the origin, which is a change only of notational nature. Its proof consists on a comparison argument, which is the content of Proposition 3.3. Afterwards, in Lemma 3.6, we show that the expected number of jumps over the set of slow bonds (i.e., those with rates α/n) is finite. Finally, with all that at hand, we are able to finish the proof.

We denote by $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$ the continuous time simple random walk on W that jumps from a site $z_1 \in W$ to any fixed neighbouring site $z_2 \in W$ at rate 1, i.e., the simple random walk reflected at the boundary of W (which takes a triangular shape, see Figure 3). In particular the total jump rate out of $z_1 \in W$ is equal to the number of nearest neighbours of z_1 that lay inside W . Expectation with respect to $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$ conditioned to start at $(x, y) \in W$ is denoted by $\mathbf{E}_{(x,y)}^\nabla$.

Lemma 3.2. *There exists a constant $c > 0$ such that for all $t \geq 0$ and all $(x, y) \in W$,*

$$\mathbf{E}_{(x,y)}^\nabla[L_t(\partial W)] \leq c\sqrt{t}, \tag{3.5}$$

where $\partial W = \{(x, y) \in W : x = 0, y \geq 0 \text{ or } x = y\}$.

To prove the above lemma we will need two additional results. To introduce the first one, we remind the reader that a continuous time Markov chain on a countable set \mathcal{E} can be constructed from a transition probability p on \mathcal{E} and a bounded function $\lambda : \mathcal{E} \rightarrow (0, \infty)$ as follows:

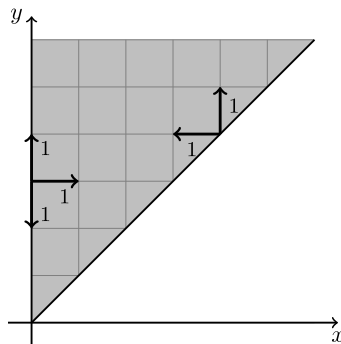


Fig. 3. For $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$ any jump rate is equal to 1.

- (1) sample a discrete time Markov chain $(\xi_n)_{n \geq 0}$ with transition probability p ;
- (2) sample a sequence of independent random variables $(\tau_n)_{n \geq 0}$ such that τ_n is exponentially distributed with rate $\lambda(\xi_n)$ and define the successive sequence of jump times via $T_0 = 0$ and $T_n = \tau_n + T_{n-1}$ for $n \geq 1$;
- (3) finally, define the continuous time Markov chain Z via

$$Z_t = \xi_n \mathbf{1}_{\{T_n \leq t < T_{n+1}\}}.$$

Note in particular that in the above construction the jump rate from a vertex x to y is given by $\zeta(x, y) = \lambda(x)p(x, y)$. In the sequel we also will assume without further mentioning that p does not put any mass on the diagonal. To continue, we fix a transition probability p on \mathcal{E} and for any a, b such that $0 < a \leq b < \infty$, we denote by $Z^{[a,b]}$ the continuous time Markov chain with transition probability p and such that its field of jump rates $(\lambda_{[a,b]}(x))_{x \in \mathcal{E}}$ is such that $\lambda_{[a,b]}(x) \in [a, b]$ for all $x \in \mathcal{E}$. We denote the expectation with respect to $Z^{[a,b]}$ started in $z \in \mathcal{E}$ by $\mathbf{E}_z^{[a,b]}$.

Proposition 3.3. Fix $0 < a < b \leq c < d < \infty$, and define $\Lambda \stackrel{\text{def}}{=} \sup_{x \in \mathcal{E}} \frac{\lambda_{[c,d]}(x)}{\lambda_{[a,b]}(x)}$. For any $A \subseteq \mathcal{E}$ and any $z \in \mathcal{E}$,

$$\mathbf{E}_z^{[c,d]}[L_t(A)] \leq \mathbf{E}_z^{[a,b]}[L_{\Lambda t}(A)]. \tag{3.6}$$

The second result is about projections (also called lumping) of continuous time Markov chains.

Proposition 3.4. Let \mathcal{E} be a countable set, and consider a bounded function $\zeta : \mathcal{E} \times \mathcal{E} \rightarrow [0, \infty)$. Let $(X_t)_{t \geq 0}$ be the continuous time Markov chain with state space \mathcal{E} and jump rates $\{\zeta(x, y)\}_{x, y \in \mathcal{E} \times \mathcal{E}}$. Fix an equivalence relation \sim on \mathcal{E} with equivalence classes $\mathcal{E}^\sharp = \{[x] : x \in \mathcal{E}\}$ and assume that ζ satisfies

$$\sum_{y' \sim y} \zeta(x, y') = \sum_{y' \sim y} \zeta(x', y') \tag{3.7}$$

whenever $x \sim x'$. Then, $([X_t])_{t \geq 0}$ is a Markov chain with state space \mathcal{E}^\sharp and jump rates $\zeta([x], [y]) = \sum_{y' \sim y} \zeta(x, y')$.

We first prove Proposition 3.3, afterwards Proposition 3.4 and finally we prove Lemma 3.2.

Proof of Proposition 3.3. To prove (3.6) we use a coupling argument. We do so by first sampling the discrete time Markov chain $(\xi_n)_{n \geq 0}$ as alluded above, and we intend to construct $Z^{[a,b]}$ and $Z^{[c,d]}$ both from the same realization of $(\xi_n)_{n \geq 0}$. To that end, we consider an independent field of Poisson clocks $(N_x^{[c,d]})_{x \in \mathcal{E}}$ such that for any $x \in \mathcal{E}$ the rate of $N_x^{[c,d]}$ equals $\lambda_{[c,d]}(x)$. We further define

$$N_x^{[a,b]}(t) \stackrel{\text{def}}{=} N_x^{[c,d]} \left(\frac{t}{\lambda_{[c,d]}(x)} \lambda_{[a,b]}(x) \right)$$

and it readily follows that for each $x \in \mathcal{E}$ the process $N_x^{[a,b]}$ is a Poisson process with rate $\lambda_{[a,b]}(x)$. Hence, it follows from the construction outlined before the statement of Proposition 3.3 that the construction above yields indeed a coupling of $Z^{[a,b]}$ and $Z^{[c,d]}$.

This coupling has the following two properties, which immediately proves (3.6). Denote by $Z_{[0,t]}^{[c,d]}$ the sequence of visited points by the process $Z^{[c,d]}$ until time t , with an analogous definition for $Z_{[0,\Lambda t]}^{[a,b]}$.

- (1) There exists some $u \in [0, \Lambda t]$ such that $Z_{[0,t]}^{[c,d]} = Z_{[0,u]}^{[a,b]}$. That is, the sequence $Z_{[0,t]}^{[c,d]}$ is an initial piece of $Z_{[0,\Lambda t]}^{[a,b]}$, so that in particular $Z_{[0,t]}^{[c,d]} \subseteq Z_{[0,\Lambda t]}^{[a,b]}$.
- (2) Given $x \in Z_{[0,t]}^{[c,d]}$, then at its k -th visit of the discrete chain $(\xi_n)_{n \in \mathbb{N}}$ to x the holding time at that point of $Z^{[a,b]}$ is larger than the one of $Z^{[c,d]}$. □

Proof of Proposition 3.4. Let P be the transition matrix of the skeleton chain of $(X_t)_{t \geq 0}$ (i.e., of the underlying discrete time Markov chain). Assumption (3.7) implies that

$$P(x, [y]) = P(x', [y])$$

whenever $x \sim x'$. It then follows from [15, Lemma 2.5, pp. 25] that the skeleton chain of $([X_t])_{t \geq 0}$ is a discrete time Markov chain with transition matrix given by $P^\sharp([x], [y]) := P(x, [y])$. Thus, it remains to show that the holding times

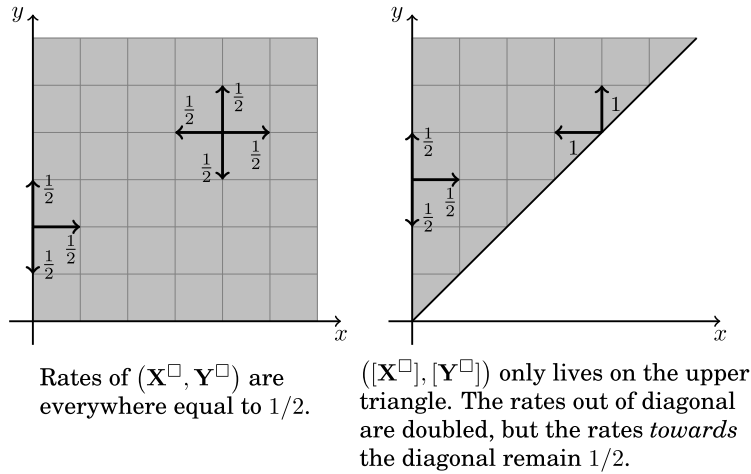


Fig. 4. Relation between $(\mathbf{X}^\square, \mathbf{Y}^\square)$ and $([\mathbf{X}^\square], [\mathbf{Y}^\square])$.

of $([X_t])_{t \geq 0}$ are exponentially distributed with rates $\{\sum_{[y]} \zeta([x], [y])\}_{[x] \in \mathcal{E}^\#}$. Yet, this is, as well, a consequence of (3.7). Hence, we can conclude the proof. \square

Proof of Lemma 3.2. The proof comes in two steps.

1st Step. In this step we show that it is sufficient to estimate the local time of a simple random walk on

$$\mathbb{Z}_{\geq 0}^2 \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : x, y \geq 0\},$$

and we refer the reader to Figures 3 and 4 for an illustration of the various random walks that will appear in this part of the proof. To that end, let $(\mathbf{X}^\square, \mathbf{Y}^\square)$ be a simple random walk defined on $\mathbb{Z}_{\geq 0}^2$ that jumps from $z_1 \in \mathbb{Z}_{\geq 0}^2$ to a fixed neighbouring site $z_2 \in \mathbb{Z}_{\geq 0}^2$ at rate $\frac{1}{2}$. Write

$$\partial W_{\text{diag}} \stackrel{\text{def}}{=} \{(x, y) \in W : x = y\} \tag{3.8}$$

and note that

$$\partial \mathbb{Z}_{\geq 0}^2 = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 : x = 0 \text{ or } y = 0\}. \tag{3.9}$$

Our aim is to show that for any $(x, y) \in W$,

$$\mathbf{E}_{(x,y)}^\square [L_t(\partial W)] \leq \mathbf{E}_{(x,y)}^\square [L_{2t}(\partial \mathbb{Z}_{\geq 0}^2)] + \mathbf{E}_{(x,y)}^\square [L_{2t}(\partial W_{\text{diag}})], \tag{3.10}$$

where the expectations on the right hand side of the display above denote the expectation with respect to $(\mathbf{X}^\square, \mathbf{Y}^\square)$ started at (x, y) . To see that (3.10) is true we consider the function $T : \mathbb{Z}_{\geq 0}^2 \rightarrow \mathbb{Z}_{\geq 0}^2$ that maps each $z \in \mathbb{Z}_{\geq 0}^2$ to its reflection with respect to the diagonal ∂W_{diag} . Note in particular that T is its own inverse, so that we can define an equivalence relation on $\mathbb{Z}_{\geq 0}^2$ via

$$z_1 \sim z_2 \iff \exists n \in \{1, 2\} \text{ such that } T^n(z_1) = z_2. \tag{3.11}$$

Writing \mathbf{P}^\square for the transition matrix corresponding to the underlying discrete time random walk of $(\mathbf{X}^\square, \mathbf{Y}^\square)$, and by $\{\zeta(x, y)\}_{x,y \in \mathbb{Z}_{\geq 0}^2}$ its field of rates, it is easy to see that for any $z_1 \sim z_2$ and any $z_3 \in \mathbb{Z}_{\geq 0}^2$

$$\sum_{z'_3 \sim z_3} \zeta(z_1, z'_3) = \sum_{z'_3 \sim z_3} \zeta(z_2, z'_3). \tag{3.12}$$

Hence, by Proposition 3.4 and the fact that the set $\mathbb{Z}_{\geq 0}^2 / \sim$ equals W , the process $([\mathbf{X}^\square], [\mathbf{Y}^\square])$ can be identified with a simple random walk on W such that its jump rate out of each fixed edge equals $\frac{1}{2}$ except for those attached to ∂W_{diag} , where the jump rate is 1. Note in particular that this makes the set of edges directed. Indeed, the jump rate from $x \in \partial W_{\text{diag}}$

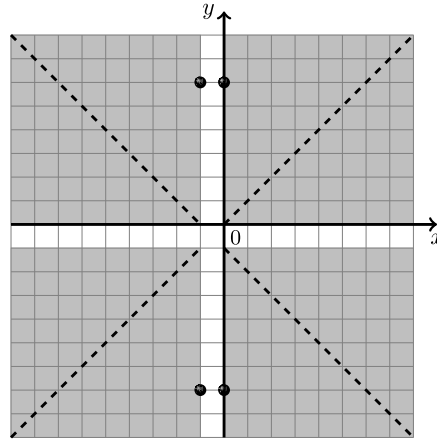


Fig. 5. Illustration of equivalence relation in the 2nd Step of the proof of Lemma 3.2. ∂W_{diag} gets identified with the points on dashed lines. The four points marked with black balls compose a single equivalence class. Non-zero jump rates between any two equivalence classes are everywhere equal to $1/2$.

to any neighbour y is 1, whereas the jump rate from y to x is $\frac{1}{2}$. Note that as a set we may identify $[\partial W]$ (the set of elements equivalent to elements in ∂W) with $\partial \mathbb{Z}_{\geq 0}^2 \cup \partial W_{\text{diag}}$. Thus, denoting by $\mathbf{E}_{([x],[y])}^{[\square]}$ the expectation with respect to $([\mathbf{X}^\square], [\mathbf{Y}^\square])$ when started at $([x], [y])$, we see that as a consequence of Proposition 3.3,

$$\mathbf{E}_{(x,y)}^\nabla [L_t(\partial W)] \leq \mathbf{E}_{([x],[y])}^{[\square]} [L_{2t}([\partial W])]. \tag{3.13}$$

Thus, (3.10) readily follows from last inequality.

2nd Step. We now show that it is sufficient to estimate certain local times of a simple random walk (X, Y) on \mathbb{Z}^2 jumping at total rate 2 (i.e., the jump rate over any fixed edge is $\frac{1}{2}$), which will then yield the claim. To that end we define an equivalence relation by imposing that $(x, y) \sim (x, -y - 1)$ and $(x, y) \sim (-x - 1, y)$, for any $x, y \in \mathbb{Z}$. We then note that in this way ∂W_{diag} gets identified with

$$[\partial W_{\text{diag}}] \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : x = y\} \cup \{(x, y) \in \mathbb{Z}^2 : x = -y - 1, y \geq 0 \text{ or } y = -x - 1, x \geq 0\},$$

see Figure 5. Note that by Proposition 3.4 the random walk $(\mathbf{X}^\square, \mathbf{Y}^\square)$ can be identified with $([X], [Y])$. This shows that it is sufficient to bound

$$\mathbf{E}_{(x,y)}^{(X,Y)} [L_{2t}(A)], \tag{3.14}$$

where $A = A_1 \cup A_2 \cup A_3$ with

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : x = y\}, \\ A_2 &\stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : x = -y - 1, y \geq 0 \text{ or } y = -x - 1, x \geq 0\} \quad \text{and} \\ A_3 &\stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : x \in \{0, -1\}, \text{ or } y \in \{0, -1\}\}. \end{aligned} \tag{3.15}$$

Since $X - Y$ has the same law as a one-dimensional symmetric simple random walk, we conclude that $L_{2t}(A_1)$ equals in law to the local time at zero of a one-dimensional symmetric simple random walk, for which the statement of this lemma is well known, and for completeness, we provide a short proof of it in Proposition A.1. A similar argument may be used for A_2 , and A_3 . Therefore, we can finish the proof. \square

Our proof immediately yields the following corollary.

Corollary 3.5. *There exists a constant $c > 0$ such that for all $t \geq 0$ and all $(x, y) \in \mathbb{Z}_{\geq 0}^2$,*

$$\mathbf{E}_{(x,y)}^{[\square]} [L_t(\partial \mathbb{Z}_{\geq 0}^2)] \leq c\sqrt{t}. \tag{3.16}$$

We now come back to the original problem, i.e., estimating local times of the random walk (\mathbf{X}, \mathbf{Y}) defined on the set V . An important ingredient in the analysis will be an estimate on the number of jumps of (\mathbf{X}, \mathbf{Y}) over the set \mathcal{U} of slow edges, i.e., those that are depicted with thick black segments in Figure 2. We define a sequence of stopping times via

$$\begin{aligned} \tau_0 &= 0, \\ \tau_1 &= \inf\{t \geq 0 : (\mathbf{X}_t, \mathbf{Y}_t) \text{ crossed an edge in } \mathcal{U}\}, \quad \text{and for } i \geq 2, \\ \tau_i &= \inf\{t \geq \tau_{i-1} : (\mathbf{X}_t, \mathbf{Y}_t) \text{ crossed an edge in } \mathcal{U}\}. \end{aligned} \tag{3.17}$$

Finally, we define the number of crossings until the time tn^2 via

$$C_{tn^2} = \sup\{i \geq 0 : \tau_i \leq tn^2\}. \tag{3.18}$$

Lemma 3.6. *There exists a constant $c > 0$ such that uniformly over all starting points $(x, y) \in V$, all $t \geq 0$, and all $n \in \mathbb{N}$,*

$$\mathbf{E}_{(x,y)}[C_{tn^2}] \leq c. \tag{3.19}$$

Proof. We first show that for all $i \geq 1$, uniformly in n

$$\inf_{(x,y)} \mathbf{P}_{(x,y)}[\tau_i - \tau_{i-1} \geq tn^2] > 0. \tag{3.20}$$

To that end, assume that (x, y) is in the first quadrant and $y \geq 2$. If (x, y) is any other point then instead of Lemma 3.2 below an application of Corollary 3.5 may be necessary. However, this is hopefully clear from the context so that we omit the details. In this case τ_1 can be interpreted as a first success of the simple random walk $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$, which with a slight abuse of notation is now considered on the set W given by the intersection of V with the first quadrant, in the following way: whenever $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$ is on a vertex z that is attached to a slow bond it realizes the following experiment: besides its three (one if the vertex is $(1, 2)$) independent Poisson clocks N_z^1, N_z^2 , and N_z^3 ringing at rate 1 that are needed for its graphical construction, it considers an additional independent Poisson clock $N_z(\alpha)$ ringing at rate α/n . We then say that the experiment is successful if $N_z(\alpha)$ rings before any of the other three clocks. It then follows from the construction that the time of the first success equals in law the time of the first jump of (\mathbf{X}, \mathbf{Y}) over a slow bond. Indeed, one may couple (\mathbf{X}, \mathbf{Y}) and $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$ such that they move together until the first time of success. Thus, using the fact that each experiment is independent of the evolution of $(\mathbf{X}^\nabla, \mathbf{Y}^\nabla)$, and that the set of vertices that are attached to \mathcal{S} is a subset of ∂W , we see that for any constant $\bar{c} \geq 0$,

$$\begin{aligned} \mathbf{P}_{(x,y)}[\tau_1 \geq tn^2] &\geq \mathbf{P}_{(x,y)}^\nabla[L_{tn^2}(\partial W) \leq \bar{c}\sqrt{tn}, \text{ all experiments are unsuccessful}] \\ &\geq \mathbf{P}_{(x,y)}^\nabla[L_{tn^2}(\partial W) \leq \bar{c}\sqrt{tn}] \cdot \mathbf{P}[\exp(\alpha/n) \geq \bar{c}\sqrt{tn}], \end{aligned} \tag{3.21}$$

where $\exp(\alpha/n)$ denotes an exponentially distributed random variable with rate α/n . The second term on the right hand side of above is equal to $e^{-\alpha\bar{c}\sqrt{t}}$, thus is strictly positive and independent of n for all choices of \bar{c} . Markov's inequality and Lemma 3.2 now imply that

$$\mathbf{P}_{(x,y)}^\nabla[L_{tn^2}(\partial W) \geq \bar{c}\sqrt{tn}] \leq c/\bar{c}, \tag{3.22}$$

where the constant c is taken from Lemma 3.2. Hence, choosing \bar{c} large enough we see that $\mathbf{P}_{(x,y)}^\nabla[L_{tn^2}(\partial W) \leq \bar{c}\sqrt{tn}]$ is strictly bounded away from zero uniformly in (x, y) and in n . With similar arguments we may derive the same statement for all $i \geq 2$. We next introduce the random variable

$$N = \inf\{i \geq 1 : \tau_i - \tau_{i-1} \geq tn^2\}. \tag{3.23}$$

Then, using the strong Markov property at time τ_{i-1} , bounding the probability of the event $\{\tau_i - \tau_{i-1} \geq tn^2\}$ by 1, and then once again using the strong Markov property at time τ_{i-2} , we can estimate for any $i \geq 1$,

$$\begin{aligned} \mathbf{P}_{(x,y)}[N = i] &\leq \mathbf{P}_{(x,y)}\left[\bigcap_{j=1}^{i-1} \{\tau_j - \tau_{j-1} < tn^2\}\right] \\ &= \mathbf{E}_{(x,y)}\left[\prod_{j=1}^{i-2} \mathbf{1}_{\{\tau_j - \tau_{j-1} < tn^2\}} \mathbf{E}_{(\mathbf{X}_{\tau_{i-2}}, \mathbf{Y}_{\tau_{i-2}})}[\mathbf{1}_{\{\tau_1 < tn^2\}}]\right]. \end{aligned} \tag{3.24}$$

Using (3.20), we see that there exists $c \in [0, 1)$ that is independent of the starting point (x, y) , such that the latter term above is bounded from above by

$$c \mathbf{E}_{(x,y)} \left[\prod_{j=1}^{i-2} \mathbf{1}_{\{\tau_j - \tau_{j-1} < tn^2\}} \right]. \tag{3.25}$$

Iterating the above procedure we can get that

$$\sup_{(x,y) \in V} \mathbf{P}_{(x,y)}[N = i] \leq c^{i-1}, \tag{3.26}$$

which in turn implies the uniform boundedness in $(x, y) \in V$ of the expectation of N . Since $C_{tn^2} \leq N$, this implies the claim. \square

We present now the proof of Proposition 3.1, and we focus first on the local time of the set $D \setminus \{(0, 1)\}$. For definiteness we assume that (\mathbf{X}, \mathbf{Y}) starts in $(x, y) \in W$, where we recall that W is a suitable shifted version of V intersected with the first quadrant. All other cases follow by a straightforward adaptation of this proof. Note that the event $\{(\mathbf{X}_s, \mathbf{Y}_s) \in D \setminus \{(0, 1)\}\}$ is only possible, if $s \in \bigcup_{i=0}^{\infty} [\tau_{2i}, \tau_{2i+1})$, where $\tau_0 = 0$. Hence, we can write

$$\begin{aligned} \mathbf{E}_{(x,y)}[L_{tn^2}(D \setminus \{(0, 1)\})] &= \mathbf{E}_{(x,y)} \left[\int_0^{tn^2} \mathbf{1}_{\{(\mathbf{X}_s, \mathbf{Y}_s) \in D \setminus \{(0, 1)\}\}} ds \right] \\ &= \sum_{i=0}^{\infty} \mathbf{E}_{(x,y)} \left[\int_{\tau_{2i} \wedge tn^2}^{\tau_{2i+1} \wedge tn^2} \mathbf{1}_{\{(\mathbf{X}_s, \mathbf{Y}_s) \in D \setminus \{(0, 1)\}\}} ds \right]. \end{aligned} \tag{3.27}$$

Fix $i \in \mathbb{N}$. Applying the strong Markov property at time τ_{2i} we can rewrite each summand in the display above as

$$\mathbf{E}_{(x,y)} \left[\mathbf{1}_{\{\tau_{2i} < tn^2\}} \mathbf{E}_{(\mathbf{X}_{\tau_{2i}}, \mathbf{Y}_{\tau_{2i}})} \left[\int_0^{\bar{\tau}_1 \wedge tn^2 - \tau_{2i}} \mathbf{1}_{\{(\bar{\mathbf{X}}_s, \bar{\mathbf{Y}}_s) \in D \setminus \{(0, 1)\}\}} ds \right] \right], \tag{3.28}$$

where $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ denotes an independent copy of (\mathbf{X}, \mathbf{Y}) and $\bar{\tau}_1$ is the corresponding stopping time, defined in the same way as τ_1 in (3.17). We now recall that as a consequence of the proof of Lemma 3.6 until time $\bar{\tau}_1$ the walk (\mathbf{X}, \mathbf{Y}) can be coupled with $(\mathbf{X}^{\nabla}, \mathbf{Y}^{\nabla})$. Hence, we see that (3.28) is at most

$$\mathbf{E}_{(x,y)}[\mathbf{1}_{\{\tau_{2i} < tn^2\}}] \sup_{(x,y) \in W} \mathbf{E}_{(x,y)}^{\nabla} \left[\int_0^{tn^2} \mathbf{1}_{\{(\mathbf{X}_s^{\nabla}, \mathbf{Y}_s^{\nabla}) \in D \setminus \{(0, 1)\}\}} ds \right]. \tag{3.29}$$

Making use of Lemma 3.2 we see that there exists a constant $c \in (0, \infty)$ such that for all starting points, all $t \geq 0$ and all $n \in \mathbb{N}$, the term on the left hand-side of (3.27) is bounded from above by

$$c \sqrt{tn} \mathbf{E}_{(x,y)}[C_{tn^2}]. \tag{3.30}$$

Hence, an application of Lemma 3.6 is enough to conclude the claim. To estimate the local time of the vertex $(0, 1)$ we can proceed almost exactly as above, and we see that there exists a constant $c \in (0, +\infty)$ such that

$$\begin{aligned} \mathbf{E}_{(x,y)}[L_{tn^2}(\{(0, 1)\})] &= \int_0^{tn^2} \mathbf{P}_{(x,y)}[(\mathbf{X}_s, \mathbf{Y}_s) = (0, 1)] ds \\ &\leq c \sum_{z \in \mathcal{A}} \int_0^{2tn^2} \mathbf{P}_{(x,y)}[(X_s, Y_s) = z] ds, \end{aligned} \tag{3.31}$$

where we recall that (X, Y) denotes the simple random walk on \mathbb{Z}^2 jumping at total rate 2, and $\mathcal{A} = \{(0, 1), (1, 1), (0, 0), (1, 0)\}$. The proof now follows from the local central limit theorem, see for instance [14, Theorem 2.5.6] (this result is stated for one-dimensional continuous time random walks, however using the fact that a d -dimensional continuous time random walk consists of d independent one-dimensional random walks, it may be easily adapted to our setting), or alternatively from Proposition A.2.

3.2. Estimates in dimension one

We denote by $\{\mathbf{X}_t; t \geq 0\}$ the random walk on \mathbb{Z} with a slow bond, that is, the random walk with infinitesimal generator \mathcal{A}_n given in (2.10) and we use $\mathbf{E}_x, \mathbf{P}_x$ to denote the corresponding expectation and probability, starting from $x \in \mathbb{Z}$.

Lemma 3.7. *For all $x, y \in \mathbb{Z}$, and for all $t \geq 0$ we have the equality*

$$\mathbf{P}_x(\mathbf{X}_t = y) + \mathbf{P}_x(\mathbf{X}_t = -y + 1) = \mathbf{P}_x(X_t = y) + \mathbf{P}_x(X_t = -y + 1), \tag{3.32}$$

where $(X_t)_{t \geq 0}$ denotes a one-dimensional symmetric simple random walk jumping at total rate 2.

Proof. The proof comes in two steps.

1st Step. In this step we rewrite the left hand-side in (3.32) in terms of the transition probabilities of a symmetric simple random walk that is reflected at 1. To that end, we define the following equivalence relation:

$$x \sim y \iff y = -x + 1 \text{ or } y = x. \tag{3.33}$$

Note that in particular in this way 0 gets identified with 1, so that jumps between these two vertices “do not count”. One may then readily check that condition (3.7) is satisfied, so that $([\mathbf{X}_t])_{t \geq 0}$ defines a continuous time Markov chain. It is then plain to see that for all $x \in \mathbb{Z}$ and all $t \geq 0$ the following relation holds:

$$[\mathbf{X}_t] = [x] \iff \mathbf{X}_t \in \{x, -x + 1\}. \tag{3.34}$$

Thus,

$$\mathbf{P}_x(\mathbf{X}_t \in \{y, -y + 1\}) = \mathbf{P}_{[x]}([\mathbf{X}_t] = [y]). \tag{3.35}$$

Choosing only representants in the set $\mathbb{Z}_{\geq 1} \stackrel{\text{def}}{=} \{x \in \mathbb{Z} : x \geq 1\}$ we see that $([\mathbf{X}_t])_{t \geq 0}$ may be identified with a simple random walk $(X_t^R)_{t \geq 0}$ on $\mathbb{Z}_{\geq 1}$ that jumps from any vertex $x \in \mathbb{Z}_{\geq 1}$ to a fixed neighboring vertex in $\mathbb{Z}_{\geq 1}$ at rate 1. Thus, for any $x, y \geq 1$, (3.35) becomes

$$\mathbf{P}_x(\mathbf{X}_t \in \{y, -y + 1\}) = \mathbf{P}_x(X_t^R = y). \tag{3.36}$$

2nd Step. In this step we show that the right hand-side of (3.36) may be rewritten in terms of a symmetric simple random walk on \mathbb{Z} jumping at total rate 2. To that end we use the same equivalence relation as above and we note that $(X_t)_{t \geq 0}$ can be, in the same way, identified with $(X_t^R)_{t \geq 0}$ as $(\mathbf{X}_t)_{t \geq 0}$ can be identified with $([\mathbf{X}_t])_{t \geq 0}$. This finishes the proof. \square

4. Estimates on the discrete derivative and correlations

In the next two subsections, we present the proofs of Theorems 2.4 and 2.5, respectively.

4.1. Estimate on the discrete derivative

This section is devoted to the proof of Theorem 2.4.

Proof. Recall that ρ_t^n is a solution of (2.9). Since the statement is clear for $x = 0$, we only need to deal with the case $x \neq 0$. Let ρ_t be the solution of the equation (2.4), and define $\gamma^n : [0, T] \times \mathbb{Z}^d \rightarrow \mathbb{R}$ via

$$\gamma_t^n(x) \stackrel{\text{def}}{=} \begin{cases} \rho_t^n(x) - \rho_t(\frac{x}{n}) & \text{if } x \neq 0, \\ \rho_t^n(0) - \rho_t(\frac{-1}{n^2}) & \text{otherwise.} \end{cases} \tag{4.1}$$

The reason for the previous definition is that it distinguishes two cases, since at $x = 0$ the time derivative of ρ is not related to its spatial derivatives in a way that is helpful for our purposes. However, with the above choice of γ^n we see that for all $x \in \mathbb{Z}$,

$$\partial_t \gamma_t^n(x) = n^2 \mathcal{A}_n \gamma_t^n(x) + F_t^n(x), \tag{4.2}$$

where

$$F_t^n(x) \stackrel{\text{def}}{=} \begin{cases} (n^2 \mathcal{A}_n - \partial_u^2) \rho_t(\frac{x}{n}) & \text{if } x \neq 0, \\ n^2 \mathcal{A}_n \rho_t(0) - \partial_u^2 \rho_t(\frac{-1}{n^2}) & \text{otherwise.} \end{cases} \tag{4.3}$$

Observe that, by the definition of \mathcal{A}_n in (2.10), for $x \in \mathbb{Z} \setminus \{0, 1\}$, F_t^n accounts for the difference between the discrete and the continuous Laplacian. To continue, we add and subtract $\rho_t(\frac{x}{n})$ and $\rho_t(\frac{x+1}{n})$ to $|\rho_t^n(x+1) - \rho_t^n(x)|$ and use the triangle inequality which yields

$$|\rho_t^n(x+1) - \rho_t^n(x)| \leq |\gamma_t^n(x+1)| + |\gamma_t^n(x)| + \left| \rho_t\left(\frac{x+1}{n}\right) - \rho_t\left(\frac{x}{n}\right) \right|. \tag{4.4}$$

We first treat the rightmost term above. Since $x \mapsto \rho_t(x)$ is differentiable in any neighborhood outside of zero, and ρ_t has one sided spatial derivatives at zero, we see that

$$\left| \rho_t\left(\frac{x+1}{n}\right) - \rho_t\left(\frac{x}{n}\right) \right| = O\left(\frac{1}{n}\right).$$

Recall that $\{\mathbf{X}_t; t \geq 0\}$ denotes the random walk on \mathbb{Z} generated by \mathcal{A}_n . Applying Duhamel’s principle we see that we can write the solution of (4.2) as

$$\gamma_t^n(x) = \mathbf{E}_x \left[\gamma_0^n(\mathbf{X}_{tn^2}) + \int_0^t F_{t-s}^n(\mathbf{X}_{sn^2}) ds \right].$$

Therefore,

$$\sup_{t \leq T} \sup_{x \in \mathbb{Z}} |\gamma_t^n(x)| \leq \sup_{x \in \mathbb{Z}} |\gamma_0^n(x)| + \sup_{t \leq T} \sup_{x \in \mathbb{Z}} \left| \mathbf{E}_x \left[\int_0^t F_{t-s}^n(\mathbf{X}_{sn^2}) ds \right] \right|.$$

Since $|\gamma_0^n(x)| = |\rho_0^n(x) - \rho_0(x)|$, by Assumption (2.12) we only need to control the second term on the right hand-side of the previous expression. By Fubini’s Theorem, we see that

$$\mathbf{E}_x \left[\int_0^t F_{t-s}^n(\mathbf{X}_{sn^2}) ds \right] = \int_0^t \sum_{z \in \mathbb{Z}} \mathbf{P}_x[\mathbf{X}_{sn^2} = z] F_{t-s}^n(z) ds. \tag{4.5}$$

Since the discrete Laplacian approximates the continuous Laplacian, we conclude that $|F_t^n(x)| \leq C/n^2$ for any $x \in \mathbb{Z} \setminus \{0, 1\}$ and for any $t \geq 0$. Therefore, we can bound the absolute value of (4.5) by

$$t \frac{C}{n^2} + \int_0^t \sum_{z \in \{0,1\}} \mathbf{P}_x[\mathbf{X}_{sn^2} = z] \cdot |F_{t-s}^n(z)| ds. \tag{4.6}$$

We note now that

$$\begin{aligned} F_t^n(1) &= n^2 \left(\rho_t\left(\frac{2}{n}\right) - \rho_t\left(\frac{1}{n}\right) + \frac{\alpha}{n} \left(\rho_t\left(\frac{0}{n}\right) - \rho_t\left(\frac{1}{n}\right) \right) \right) - \partial_u^2 \rho_t\left(\frac{1}{n}\right) \\ &= n \left(n \left(\rho_t\left(\frac{2}{n}\right) - \rho_t\left(\frac{1}{n}\right) \right) + \alpha \left(\rho_t\left(\frac{0}{n}\right) - \rho_t\left(\frac{1}{n}\right) \right) \right) - \partial_u^2 \rho_t\left(\frac{1}{n}\right) \\ &= n \left(\frac{d\rho_t}{dx}(0^+) + \alpha(\rho_t(0^-) - \rho_t(0^+)) + O(1/n) \right) - \partial_u^2 \rho_t\left(\frac{1}{n}\right). \end{aligned}$$

The Robin boundary condition in the PDE (2.4) cancels the first two terms inside the parenthesis above, from what we deduce that $|F_t^n(1)| \leq C$ for any $t \geq 0$. For $z = 0$ we obtain, in a similar way, a bound of the same order. Therefore, (4.6) is bounded from above by

$$t \frac{C}{n^2} + C \int_0^t (\mathbf{P}_x[\mathbf{X}_{sn^2} = 0] + \mathbf{P}_x[\mathbf{X}_{sn^2} = 1]) ds.$$

Thus, applying Lemma 4.1 below the result follows. □

Lemma 4.1. *Let \mathbf{X} be as in Section 3.2. There exists a constant $C > 0$ such that the following estimate holds for all $t \geq 0$:*

$$\int_0^t \mathbf{P}_x[\mathbf{X}_{sn^2} \in \{0, 1\}] ds \leq \frac{C\sqrt{t}}{n}.$$

Proof. Denote the symmetric simple random walk on \mathbb{Z} jumping at rate 2 by $\{X_t; t \geq 0\}$. It is then well known that for all $t \geq 0$ the map $x \in \mathbb{Z} \mapsto \mathbf{P}_x[X_t = 0]$ is maximized at $x = 0$. Hence, Lemma 4.1 is a consequence of Lemma 3.7 together with Proposition A.1. \square

4.2. *Estimate on the correlation function*

In this section we prove Theorem 2.5. To that end, we show that the correlation function φ^n introduced in Definition 3 can be estimated from above by the local times of the random walk $\{(\mathbf{X}_t, \mathbf{Y}_t); t \geq 0\}$, introduced in Section 3.1. This is the content of Proposition 4.2. Proposition 3.1 then immediately yields the result. Given a set $A \subseteq V$, similarly as in Section 3 we denote by $L_t(A)$ the local time of $\{(\mathbf{X}_t, \mathbf{Y}_t); t \geq 0\}$ until time t in A , see (3.1).

Proposition 4.2. *There exists $C > 0$ such that*

$$\sup_{t \leq T} |\varphi_t^n(x, y)| \leq \frac{C}{n} + C \left(\frac{1}{n^2} \mathbf{E}_{(x,y)}[L_{n^2T}(D \setminus \{(0, 1)\})] + \frac{1}{n} \mathbf{E}_{(x,y)}[L_{n^2T}(\{(0, 1)\})] \right). \tag{4.7}$$

Proof. First, observe that from Kolmogorov’s forward equation, we have that

$$\partial_t \varphi_t^n(x, y) = \mathbb{E}_{\mu_n} [n^2 \mathcal{L}_n(\eta_t(x)\eta_t(y))] - \partial_t(\rho_t^n(x)\rho_t^n(y)).$$

Applying (2.1) and (2.8) and performing some long, but simple, calculations, one can deduce that φ_t^n solves the following equation:

$$\partial_t \varphi_t^n(x, y) = n^2 \mathbf{B}_n \varphi_t^n(x, y) + g_t^n(x, y),$$

where \mathbf{B}_n was defined in (3.2) and

$$g_t^n(x, y) = -(\nabla_n^+ \rho_t^n(x))^2 \left(\mathbf{1}_{\{D \setminus (0,1)\}} + \frac{\alpha}{n} \mathbf{1}_{\{(0,1)\}} \right). \tag{4.8}$$

Here, ∇_n^+ denotes the rescaled discrete right derivative which, for any function $f : \mathbb{Z} \rightarrow \mathbb{R}$, is defined via $\nabla_n^+ f(x) \stackrel{\text{def}}{=} n(f(x + 1) - f(x))$. By Duhamel’s Principle,

$$\varphi_t^n(x, y) = \mathbf{E}_{(x,y)} \left[\varphi_0^n(\mathbf{X}_{tn^2}, \mathbf{Y}_{tn^2}) + \int_0^t g_{t-s}^n(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) ds \right],$$

where $\{(\mathbf{X}_t, \mathbf{Y}_t); t \geq 0\}$ is the random walk with generator \mathbf{B}_n . In order to prove the proposition we just have to estimate the right hand-side of the last equation. We see that

$$\sup_{t \leq T} |\varphi_t^n(x, y)| \leq |\varphi_0^n(x, y)| + \sup_{t \leq T} \left| \mathbf{E}_{(x,y)} \left[\int_0^t g_{t-s}^n(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) ds \right] \right|. \tag{4.9}$$

By Assumption (2.13), the first term on the right hand-side of the last expression is bounded from above by C/n . Thus, to finish the proof we only need to estimate the rightmost term in the display above.

Applying the definition of g^n , and rewriting the expectation above in terms of transition probabilities, we see that for any $s \leq t$,

$$\begin{aligned} \mathbf{E}_{(x,y)} [g_{t-s}^n(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2})] &= \sum_{z \neq 0} [-(\nabla_n^+ \rho_{t-s}^n(z))^2] \mathbf{P}_{(x,y)} [(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (z, z + 1)] \\ &\quad + \frac{\alpha}{n} [-(\nabla_n^+ \rho_{t-s}^n(0))^2] \mathbf{P}_{(x,y)} [(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (0, 1)]. \end{aligned}$$

Consequently, for all $(x, y) \in V$, the rightmost term in (4.9) is bounded from above by

$$\int_0^t \left(S_n \mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) \in D \setminus \{(0, 1)\}] + S_{n,0} \frac{\alpha}{n} \mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (0, 1)] \right) ds, \tag{4.10}$$

where

$$S_n = \sup_{t \geq 0} \sup_{z \in \mathbb{Z} \setminus \{0\}} (\nabla_n^+ \rho_t^n(z))^2 \quad \text{and} \quad S_{n,0} = \sup_{t \geq 0} (\nabla_n^+ \rho_t^n(0))^2. \tag{4.11}$$

Recalling Theorem 2.4, we easily deduce that $S_n \leq C$ and $S_{n,0} \leq Cn^2$. Substituting (4.11) into (4.10), together with a change of variables, the result follows. \square

The proof of Theorem 2.5 is now an immediate consequence of Proposition 3.1.

4.3. Comments on the lower bound

In the usual symmetric simple exclusion process the correlation function is of order $O(\frac{1}{n})$. Since intuitively one could expect that the presence of the slow bond increases the correlation between sites which are located both on the positive half-line or both the negative half-line, the above result does not come as a total surprise.

However, for two sites x and y such that $x \leq 0 < 1 \leq y$, then at first sight it seems to be a reasonable guess that the correlations decrease, and they should be at most of order $O(\frac{1}{n})$. Yet, our proof yields the same bound as above when one restricts only to such kind of pairs of vertices (x, y) . A natural question therefore is if a matching lower bound in (2.14) holds. Since our assumptions on the initial measure do not exclude the choice of a product Bernoulli measure with constant intensity, in which case at any time $t \geq 0$ the covariance between two distinct points is zero, such a lower bound certainly cannot hold in general.

Nevertheless, we argue that there are indeed choices of the various parameters in our model for which $|\varphi_t^n(x, y)|$ is bounded from below by a constant times $\log n/n$ uniformly in $t \in [0, T]$. We will not provide all the details, yet the gaps can be filled by an adaptation of the techniques used in Section 3. We choose $\mu_n \sim \bigotimes_{x \in \mathbb{Z}} \text{Ber}(\rho_x)$, where

$$\rho_x = \begin{cases} \frac{1}{2} & \text{if } x \leq 0, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{4.12}$$

Analyzing carefully the proof of Theorem 2.5, we see that in order to establish the desired lower bound it is enough to show that there exists a constant $c > 0$ such that for all $t \in [0, T]$

$$|\rho_t^n(0) - \rho_t^n(1)| \geq c, \tag{4.13}$$

and that the rightmost local time term in (4.7) is bounded from below by a constant times $\log n$. We only provide a sketch of the argument for the former statement, the latter as mentioned above can be deduced by an application of the techniques developed in Section 3. We note that it is possible to show that

$$\rho_t^n(0) = \sum_{x \in \mathbb{Z}} \mathbb{P}_0[\mathbf{X}_t = x] \rho_0^n(x) \quad \text{and} \quad \rho_t^n(1) = \sum_{x \in \mathbb{Z}} \mathbb{P}_1[\mathbf{X}_t = x] \rho_0^n(x), \tag{4.14}$$

where \mathbf{X} denotes a random walk with generator $n^2 \mathcal{A}_n$, and for $z \in \mathbb{Z}$ we denoted by \mathbb{P}_z the distribution of \mathbf{X} when started in z . Using that by symmetry $\mathbb{P}_1[\mathbf{X}_t \geq 1] = \mathbb{P}_0[\mathbf{X}_t \leq 0]$ and $\mathbb{P}_1[\mathbf{X}_t \leq 0] = \mathbb{P}_0[\mathbf{X}_t \geq 1]$, as well as our choice of μ_n , we see that

$$\rho_t^n(0) - \rho_t^n(1) = \frac{1}{4} (\mathbb{P}_0[\mathbf{X}_t \leq 0] - \mathbb{P}_0[\mathbf{X}_t \geq 1]). \tag{4.15}$$

It is now possible to argue that a random walk that starts at zero, and that is reflected at zero has a local time of order n up to times of order n^2 at the origin. Using a coupling argument one may then show that one can choose α small enough so that the probability that \mathbf{X} , when started at 0, crosses the bond $(0, 1)$ before time Tn^2 becomes arbitrarily small. This readily yields that (4.15) is indeed strictly bounded away from zero uniformly in $t \in [0, T]$, and consequently we obtain a lower bound that matches the order of the upper bound in (2.14).

Remark 4.3. As argued above, at first sight it seems counterintuitive that $\varphi_t(x, y)$ is of order $\log n/n$ if $x \leq 0 < 1 \leq y$. Yet, an intuitive explanation for that phenomenon could be as follows: given an exclusion particle starting at $x \leq 0$, then up to time say $\frac{t}{2}n^2$ there is a strictly positive probability that it will cross the bond $\{0, 1\}$, and afterwards it will have interaction with a particle started at $y \geq 1$ of same order as if it had started at a site $x \geq 1$.

5. Proof of density fluctuations

In this section we prove Theorem 2.8. We follow the usual procedure to establish such a result, i.e., first we establish tightness of the sequence of density fields $\{\mathcal{Y}_t^n : t \in [0, T]\}_{n \in \mathbb{N}}$ and afterwards we characterize the limit. Before proceeding, we introduce in the next subsection some martingales associated with the density fluctuation field defined in (2.19).

5.1. Associated martingales

Fix a test function $f \in \mathcal{S}_\alpha(\mathbb{R})$. By Dynkin’s formula,

$$\mathcal{M}_t^n(f) := \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t (n^2 \mathcal{L}_n + \partial_s) \mathcal{Y}_s^n(f) ds \tag{5.1}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s, s \leq t)$. Our aim is to write this martingale in a more suitable form. Recall (2.2). Performing simple calculations,

$$\begin{aligned} n^2 \mathcal{L}_n \mathcal{Y}_s^n(f) &= n^2 \sum_{x \in \mathbb{Z}} \xi_{x, x+1}^n \left[\frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}} f\left(\frac{y}{n}\right) (\eta_s^{x, x+1}(y) - \rho_s^n(y)) - \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}} f\left(\frac{y}{n}\right) (\eta_s(y) - \rho_s^n(y)) \right] \\ &= \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \xi_{x, x+1}^n \left\{ \eta_s(x) \left[f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right] + \eta_s(x+1) \left[f\left(\frac{x}{n}\right) - f\left(\frac{x+1}{n}\right) \right] \right\} \\ &= \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \left\{ \xi_{x, x+1}^n \left[f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right] + \xi_{x-1, x}^n \left[f\left(\frac{x-1}{n}\right) - f\left(\frac{x}{n}\right) \right] \right\} \eta_s(x) \\ &= \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) \eta_s(x), \end{aligned}$$

where the operator \mathcal{A}_n has been defined in (2.10). Recalling (2.9) we get that

$$\partial_s \mathcal{Y}_s^n(f) = -\frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} f\left(\frac{x}{n}\right) \partial_s \rho_s^n(x) = -\frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) \rho_s^n(x). \tag{5.2}$$

Combining the previous equalities, we see that

$$\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) \bar{\eta}_s(x) ds. \tag{5.3}$$

Adding and subtracting the term $\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f) ds$, we can rewrite the martingale $\mathcal{M}_t^n(f)$ as

$$\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t \mathcal{Y}_s^n(\Delta_\alpha f) ds - R_t^n(f), \tag{5.4}$$

where

$$R_t^n(f) := \int_0^t \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \left\{ n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) - (\Delta_\alpha f)\left(\frac{x}{n}\right) \right\} \bar{\eta}_s(x) ds.$$

The next lemma allows us to control the error term $R_t^n(f)$ defined in the previous display, which is obtained by replacing the discrete operator \mathcal{A}_n defined in (2.10) by the continuous Laplacian Δ_α defined in (2.7).

Lemma 5.1. For any $f \in \mathcal{S}_\alpha(\mathbb{R})$, almost surely there exists a constant $c > 0$ such that for all $t \in [0, T]$ and all $n \in \mathbb{N}$ the estimate $|R_t^n(f)| \leq \frac{ct}{\sqrt{n}}$ holds.

Proof. We begin by splitting $R_t^n(f)$ as the sum

$$R_t^n(f) = \int_0^t \frac{1}{\sqrt{n}} \sum_{x \neq 0,1} \left\{ n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) - (\Delta_\alpha f)\left(\frac{x}{n}\right) \right\} \bar{\eta}_s(x) ds \tag{5.5}$$

$$+ \int_0^t \frac{1}{\sqrt{n}} \left\{ n^2 \mathcal{A}_n f\left(\frac{0}{n}\right) - (\Delta_\alpha f)\left(\frac{0}{n}\right) \right\} \bar{\eta}_s(0) ds \tag{5.6}$$

$$+ \int_0^t \frac{1}{\sqrt{n}} \left\{ n^2 \mathcal{A}_n f\left(\frac{1}{n}\right) - (\Delta_\alpha f)\left(\frac{1}{n}\right) \right\} \bar{\eta}_s(1) ds. \tag{5.7}$$

We begin by dealing with (5.5). Recall that $f \in \mathcal{S}_\alpha(\mathbb{R})$ and note that $|\bar{\eta}_s(x)| \leq 2$. Thus, taking advantage of the fact that for $x \notin \{0, 1\}$, the term $n^2 \mathcal{A}_n f\left(\frac{x}{n}\right)$ is the discrete Laplacian, and applying a Taylor expansion up to second order with the Lagrangian form of the remainder, we see that (5.5) is bounded by

$$\begin{aligned} & \frac{t}{\sqrt{n}} \sum_{x \neq 0,1} \left| n^2 \left\{ \left[\frac{1}{n} f'\left(\frac{x}{n}\right) + \frac{1}{2n^2} f''\left(\frac{x}{n}\right) + \frac{f'''(\vartheta^+(\frac{x}{n}))}{3!n^3} \right] \right. \right. \\ & \quad \left. \left. - \left[\frac{1}{n} f'\left(\frac{x}{n}\right) - \frac{1}{2n^2} f''\left(\frac{x}{n}\right) + \frac{f'''(\vartheta^-(\frac{x}{n}))}{3!n^3} \right] \right\} - (\Delta_\alpha f)\left(\frac{x}{n}\right) \right| \\ & = \frac{t}{\sqrt{n}} \sum_{x \neq 0,1} \left| \frac{f'''(\vartheta^+(\frac{x}{n}))}{3!n^3} - \frac{f'''(\vartheta^-(\frac{x}{n}))}{3!n^3} \right|, \end{aligned}$$

where $\vartheta^+(\frac{x}{n}) \in [\frac{x}{n}, \frac{x+1}{n}]$ and $\vartheta^-(\frac{x}{n}) \in [\frac{x-1}{n}, \frac{x}{n}]$. Since f''' is integrable, we conclude that (5.5) is of order $O(tn^{-5/2})$, and vanishes as n tends to infinity. Since $\Delta_\alpha f$ is bounded, we can see that the sum of (5.6) and (5.7) is equal to

$$\int_0^t \frac{1}{\sqrt{n}} \left\{ n^2 \mathcal{A}_n f\left(\frac{0}{n}\right) \right\} \bar{\eta}_s(0) ds + \int_0^t \frac{1}{\sqrt{n}} \left\{ n^2 \mathcal{A}_n f\left(\frac{1}{n}\right) \right\} \bar{\eta}_s(1) ds$$

plus a term of order $O(\frac{t}{\sqrt{n}})$. Applying the definition of \mathcal{A}_n , the expression above is equal to

$$\begin{aligned} & \int_0^t \frac{n^2}{\sqrt{n}} \left\{ \frac{\alpha}{n} \left(f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right) \right) + \left(f\left(\frac{-1}{n}\right) - f\left(\frac{0}{n}\right) \right) \right\} \bar{\eta}_s(0) ds \\ & + \int_0^t \frac{n^2}{\sqrt{n}} \left\{ \frac{\alpha}{n} \left(f\left(\frac{0}{n}\right) - f\left(\frac{1}{n}\right) \right) + \left(f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) \right) \right\} \bar{\eta}_s(1) ds, \end{aligned}$$

and we can see that the absolute value of expression above is bounded by

$$\begin{aligned} & t\sqrt{n} \left\{ \left| \alpha \left(f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right) \right) + n \left(f\left(\frac{-1}{n}\right) - f\left(\frac{0}{n}\right) \right) \right| \right. \\ & \quad \left. + \left| \alpha \left(f\left(\frac{0}{n}\right) - f\left(\frac{1}{n}\right) \right) + n \left(f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) \right) \right| \right\}. \tag{5.8} \end{aligned}$$

Since $f \in \mathcal{S}_\alpha(\mathbb{R})$, we have the boundary conditions $\alpha(f(0^+) - f(0^-)) = \partial_u f(0^+) = \partial_u f(0^-)$ and also that f is left continuous at zero, hence

$$\begin{aligned} f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right) &= [f(0^+) - f(0^-)] + O(1/n), \\ n \left[f\left(\frac{-1}{n}\right) - f\left(\frac{0}{n}\right) \right] &= -\partial_u f(0^-) + O(1/n), \\ f\left(\frac{0}{n}\right) - f\left(\frac{1}{n}\right) &= -[f(0^+) - f(0^-)] + O(1/n), \end{aligned}$$

$$n \left[f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) \right] = \partial_u f(0^+) + O(1/n),$$

which permits to conclude that (5.8) is of order $O(\frac{1}{\sqrt{n}})$, finishing the proof. □

Now we study the convergence of the sequence of martingales $\{\mathcal{M}_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$. This is the content of the next lemma.

Lemma 5.2. *For any $f \in \mathcal{S}_\alpha(\mathbb{R})$, the sequence of martingales $\{\mathcal{M}_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$ converges in distribution under the topology of $\mathcal{D}([0, T], \mathbb{R})$, as $n \rightarrow \infty$, to a mean-zero Gaussian process $\{\mathcal{M}_t(f) : t \in [0, T]\}$ of quadratic variation given by*

$$\begin{aligned} \langle \mathcal{M}(f) \rangle_t &= \int_0^t \int_{\mathbb{R}} 2\chi(\rho_s(u)) (\nabla_\alpha f(u))^2 du ds \\ &\quad + \int_0^t [\rho_s(0^-)(1 - \rho_s(0^+)) + \rho_s(0^+)(1 - \rho_s(0^-))] \nabla_\alpha f(0^+) ds. \end{aligned} \tag{5.9}$$

Proof. The proof of this lemma consists on applying [10, Theorem VIII.3.12, page 473]. According to that theorem, we have to check:

- (i) condition (3.14), defined in [10, page 474],
- (ii) condition $[\hat{\delta}_5\text{-D}]$, defined in [10, 3.4, page 470],
- (iii) condition $[\gamma_5\text{-D}]$, defined in [10, 3.3, page 470].

By [10, Assertion VIII.3.5, page 470], both conditions $[\hat{\delta}_5\text{-D}]$ and (3.14) are a consequence of

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mu_n} \left[\sup_{s \leq t} |\mathcal{M}_s^n(f) - \mathcal{M}_{s-}^n(f)| \right] = 0. \tag{5.10}$$

To show (5.10), note that only two sites of the configuration η change its values when a jump occurs. Therefore,

$$\sup_{s \leq t} |\mathcal{M}_s^n(f) - \mathcal{M}_{s-}^n(f)| = \sup_{s \leq t} |\mathcal{Y}_s^n(f) - \mathcal{Y}_{s-}^n(f)| \leq \frac{2\|f\|_\infty}{\sqrt{n}},$$

leading to (5.10). It remains to check Condition $[\gamma_5\text{-D}]$, i.e., the convergence in probability of the quadratic variation of $\mathcal{M}_t(f)$, which is given by

$$\langle \mathcal{M}^n(f) \rangle_t = \int_0^t n^2 [\mathcal{L}_n \mathcal{Y}_s^n(f)^2 - 2\mathcal{Y}_s^n(f) \mathcal{L}_n \mathcal{Y}_s^n(f)] ds.$$

After some elementary computations, the right hand-side of the display above can be rewritten as

$$\begin{aligned} &\int_0^t \frac{1}{n} \sum_{x \neq 0} (\eta_s(x) - \eta_s(x+1))^2 \left[n \left(f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right) \right]^2 ds \\ &\quad + \alpha \int_0^t (\eta_s(0) - \eta_s(1))^2 \left(f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right) \right)^2 ds, \end{aligned} \tag{5.11}$$

which is an *additive functional* of the exclusion process η_t . It is almost folklore in the literature that Theorem 2.1 together with a suitable *Replacement Lemma* and standard computations yield that (5.11) converges in distribution to the right hand-side of (5.9) as $n \rightarrow \infty$. Since this is not the main issue of the proof, and since such a Replacement Lemma under the slow bond’s presence has been studied in previous works (as in [3, Lemma 5.4] for instance), we do not present the proof of this result with full details, but only a sketch instead.

By a *Replacement Lemma* we mean a result allowing to replace the time integral of the occupation number $\eta_t(x)$ by an average on a box around x . The only difference with respect to the usual Replacement Lemma (see [13]), is the fact that we should avoid an intersection between this box and the slow bond in our setting. Hence, we define

$$\eta^\ell(x) = \begin{cases} \frac{1}{\ell} \sum_{y=x}^{x+\ell-1} \eta(y) & \text{for } x \geq 1, \\ \frac{1}{\ell} \sum_{y=x-\ell+1}^x \eta(y) & \text{for } x \leq 0, \end{cases}$$

which is related to the side limits appearing in (5.9). Taking into account these definitions, the fact that $\eta_t(x)^2 = \eta_t(x)$, and the boundary condition of f at zero, one can show that the limit in distribution of (5.11) is in fact the right hand-side of (5.9).

Since the right hand-side of (5.9) is deterministic, the convergence in distribution implies the convergence in probability, and this finishes the proof of the lemma. \square

5.2. Tightness

Let \mathcal{S} be a Fréchet space (see [18] for a definition of a Fréchet space) and denote by \mathcal{S}' its topological dual. We cite here the following useful criterion:

Proposition 5.3 (Mitoma’s criterion, [16]). *A sequence of processes $\{x_t; t \in [0, T]\}_{n \in \mathbb{N}}$ in $\mathcal{D}([0, T], \mathcal{S}')$ is tight with respect to the Skorohod topology if, and only if, the sequence $\{x_t(f); t \in [0, T]\}_{n \in \mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$, for any $f \in \mathcal{S}$.*

Since $\mathcal{S}_\alpha(\mathbb{R})$ is a Fréchet space (see [4]), tightness of the density field is reduced to showing tightness of a family of real-valued processes. For that purpose, let $f \in \mathcal{S}_\alpha(\mathbb{R})$. Since the sum of tight processes is also tight, in order to prove tightness of $\{\mathcal{Y}_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$ it is enough to prove tightness of the remaining processes appearing in (5.4), namely $\{\mathcal{Y}_0^n(f)\}_{n \in \mathbb{N}}$, $\{\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f) ds : t \in [0, T]\}_{n \in \mathbb{N}}$, $\{\mathcal{M}_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$ and $\{R_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$. We deal with all of them separately.

Observe that

$$\mathbb{E}_{\nu_{\rho_0}^n(\cdot)}[(\mathcal{Y}_0^n(f))^2] = \frac{1}{n} \sum_{x \in \mathbb{Z}} f^2\left(\frac{x}{n}\right) \chi(\rho_0^n(x)) + \frac{2}{n} \sum_{x < y} f\left(\frac{x}{n}\right) f\left(\frac{y}{n}\right) \varphi_0^n(x, y)$$

is bounded. As a consequence of Assumption (B) in Theorem 2.8 the sequence of initial conditions \mathcal{Y}_0^n converges, therefore it is also tight.

By Lemma 5.1, the sequence of processes $\{R_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$ is negligible, thus it is tight.

By Lemma 5.2 the sequence of martingales $\{\mathcal{M}_t^n(f) : t \in [0, T]\}_{n \in \mathbb{N}}$ converges, hence it is tight as well.

It remains to prove tightness of the integral terms $\{\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f) ds : t \in [0, T]\}_{n \in \mathbb{N}}$. At this point we invoke *Aldous’ criterion*:

Proposition 5.4 (Aldous’ criterion). *A sequence $\{x_t^n : t \in [0, T]\}_{n \in \mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$ if:*

- (i) $\lim_{A \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}(\sup_{0 \leq t \leq T} |x_t^n| > A) = 0$,
- (ii) for any $\varepsilon > 0$, $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} \sup_{\lambda \leq \delta} \sup_{\tau \in \mathcal{T}_T} \mathbb{P}(|x_{\tau+\lambda}^n - x_\tau^n| > \varepsilon) = 0$,

where \mathcal{T}_T is the set of stopping times bounded by T .

We first check the first item of Aldous’ criterion. By the Cauchy–Schwarz inequality,

$$\mathbb{E}_{\mu^n} \left[\sup_{t \leq T} \left(\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f) ds \right)^2 \right] \leq T \int_0^T \mathbb{E}_{\mu^n} \left[\left(\frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \Delta_\alpha f\left(\frac{x}{n}\right) (\eta_s(x) - \rho_s^n(x)) \right)^2 \right] ds.$$

Observe that the right hand-side of the display above is bounded by T^2 times

$$\frac{1}{n} \sum_{x \in \mathbb{Z}} \left(\Delta_\alpha f\left(\frac{x}{n}\right) \right)^2 \sup_{t \leq T} \chi(\rho_t^n(x)) + \frac{2}{n} \sum_{x < y} \Delta_\alpha f\left(\frac{x}{n}\right) \Delta_\alpha f\left(\frac{y}{n}\right) \sup_{t \leq T} \varphi_t^n(x, y), \tag{5.12}$$

where $\chi(\rho_t^n(x))$ was defined above (2.15) and $\varphi_t^n(x, y)$ is given in Definition 3. Since $f \in \mathcal{S}_\alpha(\mathbb{R})$, the first term in (5.12) may be easily shown to be bounded in n . As for the second term the estimate provided by Theorem 2.5 is unfortunately not quite enough. Yet, Proposition 4.2 in combination with Proposition 3.1 show that for some constants $c_1, c_2 > 0$ that do not depend on t , and (x, y) we have that for all $n \in \mathbb{N}$,

$$\varphi_t^n(x, y) \leq \frac{c_1}{n} + \frac{c_2}{n} \int_0^{Tn^2} \mathbf{P}_{(x,y)}[(\mathbf{X}_s, \mathbf{Y}_s) = (0, 1)] ds, \tag{5.13}$$

where $\{(X_t, Y_t); t \geq 0\}$ is defined in Section 3.1. Plugging the first term on the right hand-side of the display above into the second term in (5.12) gives the desired estimate. To deal with the second term on the right hand-side of (5.13) we use the fact that by (3.31) we can estimate the integral term from above by

$$c \sum_{z \in \mathcal{A}} \int_0^{2Tn^2} \mathbb{P}_{(x,y)}[(X_s, Y_s) = z] ds, \tag{5.14}$$

where (X, Y) denotes simple random walk on \mathbb{Z}^2 jumping at total rate 2, \mathcal{A} denotes the set $\{(0, 1), (1, 1), (0, 0), (1, 0)\}$, and $c \in (0, +\infty)$ is some constant. Plugging this into the second term in (5.12), and using the reversibility of (X, Y) we see that we obtain a term that is bounded from above by a constant times

$$\frac{1}{n^2} \sum_{z \in \mathcal{A}} \int_0^{2Tn^2} \mathbb{E}_z \left[\left| \Delta_\alpha f \left(\frac{X_s}{n} \right) \Delta_\alpha f \left(\frac{Y_s}{n} \right) \right| \right] ds. \tag{5.15}$$

Since $|\Delta_\alpha f(\frac{x}{n}) \Delta_\alpha f(\frac{y}{n})|$ is uniformly bounded in x and y we finally obtain that (5.12) is bounded by a constant, which implies condition (i) of Aldous' criterion via Chebychev's inequality.

We now check (ii). For this purpose, fix a stopping time $\tau \in \mathcal{T}_T$. By Chebychev's inequality and repeating the argument above, we have that

$$\mathbb{P}_{\mu^n} \left(\left| \int_\tau^{\tau+\lambda} \mathcal{Y}_s^n(\Delta_\alpha f) ds \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E}_{\mu^n} \left[\left(\int_\tau^{\tau+\lambda} \mathcal{Y}_s^n(\Delta_\alpha f) ds \right)^2 \right] \leq \frac{\delta^2 c}{\varepsilon^2},$$

which vanishes as $\delta \rightarrow 0$, and yields tightness of the integral term, and concludes therefore the proof.

5.3. Uniqueness of the Ornstein–Uhlenbeck process

The existence of the Ornstein–Uhlenbeck process solution of (2.18) is a consequence of tightness proved in Section 5.2. This subsection is devoted to the proof of uniqueness of this process, as stated in Proposition 2.7. The guideline is mainly inspired by [9,13].

In the proof of Proposition 2.7 we make use of the following result, which is a standard fact about local martingales.

Proposition 5.5. *If M_t is a local martingale with respect to a filtration \mathcal{F}_t and*

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |M_s| \right] < +\infty \tag{5.16}$$

for any $t \geq 0$, then M_t is a martingale.

Proof. Let τ_n be a sequence of stopping times such that $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ and such that the stopped process $(M_{t \wedge \tau_n})_{t \geq 0}$ is a martingale for each n . Let $s < t$, it then follows that for any $A \in \mathcal{F}_s$,

$$\mathbb{E}[M_{t \wedge \tau_n} \mathbf{1}_A] = \mathbb{E}[M_{s \wedge \tau_n} \mathbf{1}_A].$$

Letting $n \rightarrow \infty$, using (5.16) and the Dominated Convergence Theorem, we conclude that

$$\mathbb{E}[M_t \mathbf{1}_A] = \mathbb{E}[M_s \mathbf{1}_A],$$

thus finishing the proof. □

Proof of Proposition 2.7. Fix $f \in \mathcal{S}_\alpha(\mathbb{R})$ and $s > 0$. Recall the definition of the martingales $\mathcal{M}_t(f)$ and $\mathcal{N}_t(f)$ given in (2.16) and (2.17), respectively.

We claim that the process $\{X_t^s(f) : t \geq s\}$ defined by

$$X_t^s(f) = \exp \left\{ \frac{1}{2} \int_s^t \|\nabla_\alpha f\|_{\rho_r(\cdot)}^2 dr + i \left(\mathcal{Y}_t(f) - \mathcal{Y}_s(f) - \int_s^t \mathcal{Y}_r(\Delta_\alpha f) dr \right) \right\}$$

is a (complex) martingale. By [19, pp. 148, Proposition 3.4] it is immediate that $X_t^s(f)$ is a local martingale. Therefore, if we show that

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |X_u^s(f)| \right] < +\infty, \quad (5.17)$$

then, by Proposition 5.5, we conclude that $X_t^s(f)$ is a martingale. But (5.17) is a simple consequence of the fact that the function $t \mapsto \frac{1}{2} \int_0^t \|\nabla_\alpha f\|_{\rho_s(\cdot)}^2 ds$ is continuous, hence bounded on compact sets. Therefore, the claim is proved.

Fix $S > 0$. We claim now that the process $\{Z_t : 0 \leq t \leq S\}$ defined by

$$Z_t(f) = \exp \left\{ \frac{1}{2} \int_0^t \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_r(\cdot)}^2 dr + i \mathcal{Y}_t(T_{S-t}^\alpha f) \right\}$$

is also a martingale. To prove this second claim, consider two times $0 \leq t_1 < t_2 \leq S$ and a partition of the interval $[t_1, t_2]$ in n intervals of equal size, that is, $t_1 = s_0 < s_1 < \dots < s_n = t_2$, with $s_{j+1} - s_j = (t_2 - t_1)/n$. Observe now that

$$\begin{aligned} \prod_{j=0}^{n-1} X_{s_{j+1}}^{s_j} (T_{S-s_j}^\alpha f) &= \exp \left\{ \sum_{j=0}^{n-1} \frac{1}{2} \int_{s_j}^{s_{j+1}} \|\nabla_\alpha T_{S-s_j}^\alpha f\|_{\rho_s(\cdot)}^2 ds \right. \\ &\quad \left. + i \sum_{j=0}^{n-1} \left(\mathcal{Y}_{s_{j+1}}(T_{S-s_j}^\alpha f) - \mathcal{Y}_{s_j}(T_{S-s_j}^\alpha f) - \int_{s_j}^{s_{j+1}} \mathcal{Y}_r(\Delta_\alpha T_{S-s_j}^\alpha f) dr \right) \right\}. \end{aligned}$$

Due to smoothness of $T_t^\alpha f$, the first sum in the exponential above converges to

$$\frac{1}{2} \int_{t_1}^{t_2} \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_s(\cdot)}^2 dr,$$

as $n \rightarrow +\infty$. The second sum inside the exponential is the same as

$$\mathcal{Y}_{t_2}(T_{S-t_2+\frac{1}{n}}^\alpha f) - \mathcal{Y}_{t_1}(T_{S-t_1}^\alpha f) + \sum_{j=1}^{n-1} \left(\mathcal{Y}_{s_j}(T_{S-s_{j-1}}^\alpha f - T_{S-s_j}^\alpha f) - \int_{s_j}^{s_{j+1}} \mathcal{Y}_r(\Delta_\alpha T_{S-s_j}^\alpha f) dr \right).$$

Since $\mathcal{Y} \in \mathcal{C}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$, since $T_t^\alpha f$ is continuous in time and applying the expansion $T_{t+\varepsilon}^\alpha f - T_t^\alpha f = \varepsilon \Delta_\alpha T_t^\alpha f + o(\varepsilon)$, one can show that the almost sure limit of the previous expression is $\mathcal{Y}_{t_2}(T_{S-t_2}^\alpha f) - \mathcal{Y}_{t_1}(T_{S-t_1}^\alpha f)$, see [4,6] for more details. We have henceforth deduced that

$$\lim_{n \rightarrow +\infty} \prod_{j=0}^{n-1} X_{s_{j+1}}^{s_j} (T_{S-s_j}^\alpha f) = \exp \left\{ \frac{1}{2} \int_{t_1}^{t_2} \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_s(\cdot)}^2 dr + i (\mathcal{Y}_{t_2}(T_{S-t_2}^\alpha f) - \mathcal{Y}_{t_1}(T_{S-t_1}^\alpha f)) \right\} = \frac{Z_{t_2}}{Z_{t_1}}.$$

Since the complex exponential is bounded, the Dominated Convergence Theorem ensures also the convergence in L^1 . Thus,

$$\mathbb{E} \left[g \frac{Z_{t_2}}{Z_{t_1}} \right] = \lim_{n \rightarrow +\infty} \mathbb{E} \left[g \prod_{j=0}^{n-1} X_{s_{j+1}}^{s_j} (T_{S-s_j}^\alpha f) \right],$$

for any bounded function g . Take g bounded and \mathcal{F}_{t_1} -measurable. For any $f \in \mathcal{S}_\alpha(\mathbb{R})$, the process $X_t^s(f)$ is a martingale. Thus, taking the conditional expectation with respect to $\mathcal{F}_{s_{n-1}}$, we get

$$\mathbb{E} \left[g \prod_{j=0}^{n-1} X_{s_{j+1}}^{s_j} (T_{S-s_j}^\alpha f) \right] = \mathbb{E} \left[g \prod_{j=0}^{n-2} X_{s_{j+1}}^{s_j} (T_{S-s_j}^\alpha f) \right].$$

By induction, we conclude that

$$\mathbb{E} \left[g \frac{Z_{t_2}}{Z_{t_1}} \right] = \mathbb{E}[g],$$

for any bounded and \mathcal{F}_t -measurable function g . This assures that $\{Z_t : t \geq 0\}$ is a martingale. From $\mathbb{E}[Z_t | \mathcal{F}_s] = Z_s$, we get

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_r(\cdot)}^2 dr + i\mathcal{Y}_t(T_{S-t}^\alpha f)\right\} | \mathcal{F}_s\right] = \exp\left\{\frac{1}{2}\int_0^s \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_r(\cdot)}^2 dr + i\mathcal{Y}_s(T_{S-s}^\alpha f)\right\},$$

which leads to

$$\mathbb{E}\left[\exp\{i\mathcal{Y}_t(T_{S-t}^\alpha f)\} | \mathcal{F}_s\right] = \exp\left\{-\frac{1}{2}\int_s^t \|\nabla_\alpha T_{S-r}^\alpha f\|_{\rho_r(\cdot)}^2 dr + i\mathcal{Y}_s(T_{S-s}^\alpha f)\right\}.$$

Choosing $S = t$ and replacing f by λf , we achieve

$$\mathbb{E}\left[\exp\{i\lambda\mathcal{Y}_t(f)\} | \mathcal{F}_s\right] = \exp\left\{-\frac{\lambda^2}{2}\int_s^t \|\nabla_\alpha T_{t-r}^\alpha f\|_{\rho_r(\cdot)}^2 dr + i\lambda\mathcal{Y}_s(T_{t-s}^\alpha f)\right\},$$

meaning that, conditionally to \mathcal{F}_s , the random variable $\mathcal{Y}_t(f)$ has Gaussian distribution of mean $\mathcal{Y}_s(T_{t-s}^\alpha f)$ and variance $\int_s^t \|\nabla_\alpha T_r^\alpha f\|_{\rho_r(\cdot)}^2 dr$.

We claim now that this last result implies the uniqueness of the finite dimensional distributions of the process $\{\mathcal{Y}_t(f) : t \in [0, T]\}$. For the sake of clarity, consider only two times, $t_0 = 0$ and $t_1 > 0$, two test functions $f_0, f_1 \in \mathcal{S}_\alpha(\mathbb{R})$ and two Lebesgue measurable sets A_0 and A_1 . By conditioning,

$$\mathbb{P}[\mathcal{Y}_{t_1}(f_1) \in A_1, \mathcal{Y}_{t_0}(f_0) \in A_0] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{\mathcal{Y}_{t_1}(f_1) \in A_1} | \mathcal{F}_0] \cdot \mathbf{1}_{\mathcal{Y}_{t_0}(f_0) \in A_0}]].$$

Since the conditional expectation $\mathbb{E}[\mathbf{1}_{\mathcal{Y}_{t_1}(f_1) \in A_1} | \mathcal{F}_0]$ is a function of $\mathcal{Y}_{t_0}(f_1)$ and \mathcal{Y}_{t_0} is uniquely distributed as a random element of $\mathcal{S}'_\alpha(\mathbb{R})$ (by assumption (ii) of Proposition 2.7), we get that the distribution of the vector $(\mathcal{Y}_{t_1}(f_1), \mathcal{Y}_{t_0}(f_0))$ is also uniquely distributed. The generalization for a general finite number of times is straightforward.

This proves the claim, implying the uniqueness in law of the random element \mathcal{Y} and hence finishing the proof. \square

5.4. Characterization of limit points

From the results of the previous subsection we know that the sequence $\{\mathcal{Y}_t^n : t \in [0, T]\}_{n \in \mathbb{N}}$ has limit points. Let $\{\mathcal{Y}_t : t \in [0, T]\}$ be the limit in distribution of $\{\mathcal{Y}_t^n : t \in [0, T]\}_{n \in \mathbb{N}}$ along some subsequence n_k considering the uniform topology of $\mathcal{D}([0, T], \mathcal{S}'_\alpha(\mathbb{R}))$. Abusing of notation, we denote this subsequence simply by n . Our goal here is to prove that $\{\mathcal{Y}_t : t \in [0, T]\}$ satisfies the conditions (i) and (ii) of Proposition 2.7. Since assumption (B) of Theorem 2.8 gives us exactly the condition (ii), it only remains to prove condition (i). A remark: the reader may be asking about the *existence* of initial measures satisfying the assumptions (A) and (B) of Theorem 2.8. An example of such measures is postponed to Proposition B.1 in Appendix B.

For $f \in \mathcal{S}_\alpha(\mathbb{R})$, let \mathcal{M}_t and \mathcal{N}_t be the processes defined by

$$\mathcal{M}_t(f) = \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s(\Delta_\alpha f) ds,$$

$$\mathcal{N}_t(f) = (\mathcal{M}_t(f))^2 - \int_0^t \|\nabla_\alpha f\|_{\rho_s(\cdot)}^2 ds.$$

Since \mathcal{Y}_t^n is assumed to converge in distribution to \mathcal{Y}_t as $n \rightarrow +\infty$, by (5.4) and Lemma 5.1, we conclude that $\mathcal{M}_t(f)$ defined above coincides with the limit of $\mathcal{M}_t^n(f)$ as in Lemma 5.2, which was denoted by $\mathcal{M}_t(f)$ as well.

By Lemma 5.2, we already know that $\mathcal{M}_t(f)$ has quadratic variation given by $\int_0^t \|\nabla_\alpha f\|_{\rho_s(\cdot)}^2 ds$. Therefore, if we show that $\mathcal{M}_t(f)$ is a martingale, then we will immediately get that $\mathcal{N}_t(f)$ is also a martingale.

Hence, we claim that $\mathcal{M}_t(f)$ is a martingale. First of all, we fix the filtration, which will be the natural one: $\mathcal{F}_t = \{\sigma(\mathcal{Y}_s(g)) : s \leq t \text{ and } g \in \mathcal{S}_\alpha(\mathbb{R})\}$. Thus, $\mathcal{M}_t(f)$ is \mathcal{F}_t -measurable. The fact that $\mathcal{M}_t(f)$ is in L^1 for any time $t \in [0, T]$ is a consequence that $\mathcal{M}_t(f)$ is a Gaussian process, which was proved in Lemma 5.2. Thus, if we prove that

$$\mathbb{E}[\mathcal{M}_t(f)\mathbf{1}_U] = \mathbb{E}[\mathcal{M}_s(f)\mathbf{1}_U], \quad \forall U \in \mathcal{F}_s, \quad (5.18)$$

we will conclude that $\mathcal{M}_t(f)$ is a martingale. To assure (5.18) it is enough to verify it for sets U of the form

$$U = \bigcap_{i=1}^k [\mathcal{Y}_{s_i}(f_i) \in A_i]$$

for $0 \leq s_1 \leq \dots \leq s_k \leq s$, $f_i \in \mathcal{S}_\alpha(\mathbb{R})$ and A_i measurable sets of \mathbb{R} . Since $\mathcal{M}_t^n(f)$ is a martingale,

$$\mathbb{E}[\mathcal{M}_t^n(f)\mathbf{1}_{U_n}] = \mathbb{E}[\mathcal{M}_s^n(f)\mathbf{1}_{U_n}], \quad \forall U \in \mathcal{F}_s, \tag{5.19}$$

where

$$U_n = \bigcap_{i=1}^k [\mathcal{Y}_{s_i}^n(f_i) \in A_i]$$

for $0 \leq s_1 \leq \dots \leq s_k \leq s$, $f_i \in \mathcal{S}_\alpha(\mathbb{R})$ and A_i are measurable sets of \mathbb{R} . Therefore, in order to show (5.18) it is enough to prove the claim that the expectations in (5.19) converge to the respective expectations in (5.18).

Since $\mathcal{Y}_t^n(f)$ converges to $\mathcal{Y}_t(f)$ as $n \rightarrow +\infty$, which is concentrated on continuous paths, then $\mathcal{M}_t^n(f)\mathbf{1}_{U_n}$ converges in distribution to $\mathcal{M}_t(f)\mathbf{1}_U$. Thus, by [1, pp. 32, Theorem 5.4] in order to get convergence of expectations, it is enough to assure that $\{\mathcal{M}_t^n(f)\mathbf{1}_{U_n}\}_{n \in \mathbb{N}}$ is a uniformly integrable sequence. In its hand, the uniform integrability can be guaranteed by showing that the L^2 norm of $\mathcal{M}_t^n(f)\mathbf{1}_{U_n}$ is uniformly bounded in $n \in \mathbb{N}$. Since the indicator function is bounded by one, we can deal only with the L^2 norm of the martingale $\mathcal{M}_t^n(f)$. Now, applying the Minkowski inequality to (5.3), we get

$$\begin{aligned} \mathbb{E}_{\mu_n}[(\mathcal{M}_t^n(f))^2]^{1/2} &\leq \mathbb{E}_{\mu_n}[(\mathcal{Y}_t^n(f))^2]^{1/2} + \mathbb{E}_{\mu_n}[(\mathcal{Y}_0^n(f))^2]^{1/2} \\ &\quad + \mathbb{E}_{\mu_n} \left[\left(\int_0^t \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} n^2 \mathcal{A}_n f\left(\frac{x}{n}\right) \bar{\eta}_s(x) ds \right)^2 \right]^{1/2}. \end{aligned} \tag{5.20}$$

The first term on the right hand-side of (5.20) is bounded by

$$\frac{1}{n} \sum_{x \in \mathbb{Z}} \left(f\left(\frac{x}{n}\right) \right)^2 \chi(\rho_t^n(x)) + \frac{2}{n} \sum_{x < y} f\left(\frac{x}{n}\right) f\left(\frac{y}{n}\right) \varphi_t^n(x, y).$$

Since $|\rho_t^n(x)| \leq 1$, the first parcel in the display above is uniformly bounded in n . To treat the second term of the last display, we use a similar argument to the one used below (5.12). The second term on the RHS of (5.20) is bounded by

$$\frac{1}{n} \sum_{x \in \mathbb{Z}} \left(f\left(\frac{x}{n}\right) \right)^2 \chi(\rho_0^n(x)) + \frac{2}{n} \sum_{x < y} f\left(\frac{x}{n}\right) f\left(\frac{y}{n}\right) \varphi_0^n(x, y),$$

which is uniformly bounded on $n \in \mathbb{N}$ due to conditions (2.12) and (2.13). Again by a similar argument to the one presented for tightness below (5.12), the third term on the right hand-side of (5.20) is bounded by t^2 times

$$\frac{1}{n} \sum_{x \in \mathbb{Z}} \left(f\left(\frac{x}{n}\right) \right)^2 \sup_{t \leq T} \chi(\rho_t^n(x)) + \frac{2}{n} \sum_{x < y} f\left(\frac{x}{n}\right) f\left(\frac{y}{n}\right) \sup_{t \leq T} \varphi_t^n(x, y),$$

thus concluding the characterization of limit points.

Appendix A: Auxiliary results on random walks

The next result is quite classical, but hard to find in the literature. It is included here for sake of completeness.

Proposition A.1. *Let X be the symmetric simple one-dimensional continuous time random walk with jump rates given by $\lambda_{x,y} = \lambda > 0$ whenever $|x - y| = 1$, and zero if $|x - y| \neq 1$. Then,*

$$\int_0^t \mathbb{P}[X_s = 0] ds \leq c\sqrt{t},$$

where $c > 0$ is a constant which does not depend on t .

Proof. Assume for sake of simplicity that $\lambda = 1$. Let $N := N_{2s}$ be a Poisson distribution with parameter $2s$.

$$\begin{aligned}
\mathbb{P}[X_s = 0] &= \sum_{k=0}^{\infty} \mathbb{P}[X_s = 0 | N = k] \cdot \mathbb{P}[N = k] \\
&= \sum_{k=0}^{\infty} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \binom{k}{k/2} \mathbb{P}[N = k] \\
&= e^{-2s} + \sum_{k=1}^{\lfloor s \rfloor} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \binom{k}{k/2} \mathbb{P}[N = k] \\
&\quad + \sum_{k=\lfloor s \rfloor+1}^{\infty} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \binom{k}{k/2} \mathbb{P}[N = k].
\end{aligned} \tag{A.1}$$

Using the Stirling Formula (see for example Feller, Vol I.), it is easy to check that

$$\frac{1}{2^k} \binom{k}{k/2} \leq \frac{1}{\sqrt{\pi k}} \leq 1. \tag{A.2}$$

Applying the second inequality of (A.2) in the first sum of (A.1) and the first inequality of (A.2) in the second sum in (A.1), we obtain that $\mathbb{P}[X_s = 0]$ is bounded from above by

$$e^{-2s} + \mathbb{P}[N \leq \lfloor s \rfloor] + \frac{c_1}{\sqrt{s}} \sum_{k=\lfloor s \rfloor+1}^{\infty} \mathbb{P}[N = k] \leq e^{-2s} + \mathbb{P}[N \leq \lfloor s \rfloor] + \frac{c_1}{\sqrt{s}}. \tag{A.3}$$

In the sequel, we will get an exponential bound $\mathbb{P}[N \leq \lfloor s \rfloor]$ by a standard large deviations technique. In this way, note that, for any $\theta > 0$,

$$\begin{aligned}
\mathbb{P}[N \leq \lfloor s \rfloor] &\leq \mathbb{P}[N \leq s] = \mathbb{E}[\mathbf{1}_{[N \leq s]} e^{\theta N} e^{-\theta N}] \leq e^{\theta s} \mathbb{E}[\mathbf{1}_{[N \leq s]} e^{-\theta N}] \\
&\leq e^{\theta s} \mathbb{E}[e^{-\theta N}] = e^{\theta s} e^{2s(e^{-\theta} - 1)} = e^{s(2e^{-\theta} - 2 + \theta)}.
\end{aligned}$$

Denote $f(\theta) = 2e^{-\theta} - 2 + \theta$ and note that f assumes its minimum at $\theta_0 = \log 2 > 0$, and $f(\theta_0) = \log 2 - 1 < 0$. Therefore, choosing $\theta = \theta_0$, we get

$$\mathbb{P}[N \leq \lfloor s \rfloor] \leq e^{s(\log 2 - 1)}.$$

Looking at (A.3) and then to (A.1), we conclude that

$$\mathbb{P}[X_s = 0] \leq e^{-2s} + e^{s(\log 2 - 1)} + \frac{c_1}{\sqrt{s}}.$$

Integrating, we get

$$\int_0^t \mathbb{P}[X_s = 0] ds \leq \int_0^t \left(e^{-2s} + e^{s(\log 2 - 1)} + \frac{c_1}{\sqrt{s}} \right) ds \leq c_2 \sqrt{t},$$

for some constant c_2 not depending on t . □

Proposition A.2. *Let (X, Y) be the symmetric simple two-dimensional continuous time random walk. Then,*

$$\int_0^t \mathbb{P}[(X_s, Y_s) = (0, 0)] ds \leq c \log t,$$

where $c > 0$ is a constant which does not depend on t .

The proof of the statement above can be adapted from the one of Proposition A.1.

Appendix B: Fluctuations at the initial time

The objective of the next result is to illustrate that any slowly varying Bernoulli product measure satisfies the condition (B) in Theorem 2.8. Since condition (A) is trivially satisfied, this shows that the slowly varying Bernoulli measure falls into the framework of this article.

Proposition B.1. *Let $\nu_{\rho_0(\cdot)}^n$ be the slowly varying Bernoulli product measure associated with a smooth profile ρ_0 . Then, \mathcal{Y}_0^n converges in distribution to \mathcal{Y}_0 , where \mathcal{Y}_0 is a mean zero Gaussian field of covariance given by*

$$\mathbb{E}[\mathcal{Y}_0(g)\mathcal{Y}_0(f)] = \int_{\mathbb{R}} \chi(\rho_0(u))g(u)f(u) du, \tag{B.1}$$

for any $f, g \in \mathcal{S}_\alpha(\mathbb{R})$.

Proof. As argued in Section 5.2, for each $f \in \mathcal{S}_\alpha(\mathbb{R})$, the sequence $\{\mathcal{Y}_0(f)\}_{n \in \mathbb{N}}$ is tight, hence $\{\mathcal{Y}_0\}_{n \in \mathbb{N}}$ is tight due to Mitoma’s criterion (Proposition 5.3). Thus, it remains only to characterize the joint limit in distribution for the vectors of the form $(\mathcal{Y}_0(f_1), \dots, \mathcal{Y}_0(f_k))$, with $f_i \in \mathcal{S}_\alpha(\mathbb{R})$, for $i = 1, \dots, k$. Since $\nu_{\rho_0}^n$ is a product measure,

$$\begin{aligned} \log \mathbb{E}_{\nu_{\rho_0(\cdot)}^n}[\exp\{i\theta \mathcal{Y}_0^n(f)\}] &= \sum_{x \in \mathbb{Z}} \log \mathbb{E}_{\nu_{\rho_0(\cdot)}^n} \left[\exp \left\{ \frac{i\theta}{\sqrt{n}} \bar{\eta}_0(x) f \left(\frac{x}{n} \right) \right\} \right] \\ &= \sum_{x \in \mathbb{Z}} \log \left[\rho_0 \left(\frac{x}{n} \right) \exp \left\{ \frac{i\theta}{\sqrt{n}} f \left(\frac{x}{n} \right) \left(1 - \rho_0 \left(\frac{x}{n} \right) \right) \right\} \right. \\ &\quad \left. + \left(1 - \rho_0 \left(\frac{x}{n} \right) \right) \exp \left\{ -\frac{i\theta}{\sqrt{n}} f \left(\frac{x}{n} \right) \rho_0 \left(\frac{x}{n} \right) \right\} \right]. \end{aligned}$$

Since $f \in \mathcal{S}_\alpha(\mathbb{R})$, we have smoothness of f except possibly at $x = 0$, together with fast decaying. Keeping this in mind, Taylor’s expansion on the exponential function permits to conclude that the expression above is equal to

$$-\frac{\theta^2}{2n} \sum_{x \in \mathbb{Z}} f^2 \left(\frac{x}{n} \right) \chi \left(\rho_0 \left(\frac{x}{n} \right) \right) + O \left(\frac{1}{\sqrt{n}} \right),$$

which gives us that

$$\lim_{n \rightarrow +\infty} \log \mathbb{E}_{\nu_{\rho_0(\cdot)}^n}[\exp\{i\theta \mathcal{Y}_0^n(f)\}] = -\frac{\theta^2}{2} \int_{\mathbb{R}} \chi(\rho_0(u))f^2(u) du.$$

Replacing f by a linear combination of functions and then applying the Crámer–Wold device, the proof ends. □

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