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LARGE DEVIATIONS FOR THE EXCLUSION PROCESS WITH A SLOW BOND

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We consider the one-dimensional symmetric simple exclusion process with a slow bond. In this model, whilst all the transition rates are equal to one, a particular bond, the *slow bond*, has associated transition rate of value N^{-1}, where N is the scaling parameter. This model has been considered in previous works on the subject of hydrodynamic limit and fluctuations. In this paper, assuming uniqueness for weak solutions of hydrodynamic equation associated to the perturbed process, we obtain dynamical large deviations estimates in the diffusive scaling. The main challenge here is the fact that the presence of the slow bond gives rise to Robin's boundary conditions in the *continuum*, substantially complicating the large deviations scenario.

1. Introduction. In this paper, we present dynamical large deviations estimates for the Symmetric Simple Exclusion Process (SSEP) with a slow bond. The SSEP is a largely studied process both in probability and statistical mechanics. It consists of particles that perform independent random walks in a certain graph, except for the exclusion rule that prevents two or more particles from occupying the same site.

The SSEP with *a slow bond* is characterized by a defect at a fixed bond. The graph here considered is $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, the discrete one-dimensional torus with N sites. Let us describe this process in terms of clocks. At each bond, we associate a different Poisson clock, all of them independent. When a clock rings, the occupation at the sites connected by the corresponding bond are exchanged. Of course, if both sites are empty or occupied, nothing happens. We call the parameters of those Poisson clocks of *exchange rates*. All exchange rates are equal to one, except at the slow bond which has exchange rate N^{-1}, which slows down the passage of particles there. Notice that the choice of the exchange rates characterizes the non-homogeneity of the environment.

This model has origin in the models considered in [2, 7]. In [2], the exchange rate at a bond of vertices x and $x+1$ is taken as $[N(W(x+1/N)-W(x/N))]^{-1}$, where W is a α-stable subordinator of a Lévy process. In the same line, [7] dealt

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with exchange rates driven by a general, nonrandom, strictly increasing function W. The SSEP with a slow bond is in fact a particular case of the model considered in [7].

In order to understand the collective behavior of the microscopic system, a natural question is the limit for the time evolution of the spatial density of particles, usually called *hydrodynamic limit*; see [9] and references therein. The limiting density of a given system is usually characterized as the weak solution of some partial differential equation, being the associated equation denominated *hydrodynamic equation*.

By [4, 6, 7], the hydrodynamic limit of the SSEP with a slow bond is well understood, being the hydrodynamic equation given by following heat equation with Robin's boundary conditions:

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u) & t>0, u \in \mathbb{T}\backslash\{0\}, \\ \partial_u \rho(t,0^+) = \partial_u \rho(t,0^-) = \rho(t,0^+) - \rho(t,0^-) & t>0, \\ \rho(0,u) = \gamma(u) & u \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ denotes the continuous one-dimensional torus, 0^+ and 0^- denote the side limits around $0 \in \mathbb{T}$ and $\gamma : \mathbb{T} \to [0,1]$ is a density profile. The boundary condition above can be interpreted as *Fick's law*: the rate in which mass is exchanged between two media is proportional to the difference of concentration in each medium.

The natural questions that emerge in the sequence are fluctuations and large deviations with respect to the expected limit. Equilibrium fluctuations for the SSEP with a slow bond has been studied in [5]. In this work, we analyze the corresponding large deviations, consisting in the occurrence rate of events differing from the expected limit in the scaling of the hydrodynamic limit. The large deviations of a Markov process come from two origins. One part are deviations from the initial measure, and the second are deviations from the dynamics. These are called statical and dynamical large deviations, respectively. Since the invariant measures for the dynamics here considered are Bernoulli product measure, for which the large deviations are well known, we will treat only the dynamical large deviations: the system will start from *deterministic* initial configurations associated in some sense (Definition 2.1) to a macroscopic profile.

The main difficulty for establishing large deviations for the SSEP with a slow bond of parameter N^{-1} comes from the fact that the limiting occupations at the vertices of the slow bond depend on time, as we can see in the Robin's boundary conditions above. In important previous papers [1] and [3], the authors have considered exclusion processes with fixed rate of incoming and outcoming particles at the boundaries leading to Dirichlet's boundary conditions, therefore, with time independent values at the boundaries.

Here, it has been considered a single slow bond. An extension to a finite number of slow bonds (in the setting of [4]) would be straightforward, with no additional obstacles. However, it would carry on the notation and probably lead to loss of clarity. For this reason, we decided to focus in the single slow bond case. What is still far from manageability are the large deviations for the model of [7], which deals with much stronger spatial nonhomogeneity (a dense set of slow bonds is allowed there). This is a very interesting and challenging problem.

An important ingredient in the large deviations proof consists in establishing the law of large numbers for a suitable set of perturbations of the original systems. The family of perturbations we have considered is *the weakly asymmetric exclusion process (WASEP) with a slow bond*. Its hydrodynamic equation is a nonlinear diffusive partial differential equation with nonlinear Robin's boundary conditions. Assuming uniqueness of weak solutions of this equation, which is a delicate question due to the nonlinearity at the boundary, we prove the corresponding hydrodynamic limit. Existence of weak solutions is granted by the tightness of the processes.

The Radon–Nikodym derivative of the perturbed process with respect to the original process naturally leads to the expression of the large deviations rate functional. A difficulty in the proof of the upper bound comes from fact the Radon–Nikodym derivative obtained is not a function of the empirical measure. To overcome this obstacle, we show that the Radon–Nikodym derivative is superexponentially close to a function of the empirical measure. Moreover, following the steps of [1, 3], we define an energy and then prove that trajectories with infinite energy are not relevant in the large deviations regime. These results enable us to invoke the Minimax lemma, which is an important device to obtain large deviations upper bound for compact sets. Exponential tightness finally leads to the upper bound for closed sets.

Since the upper bound is achieved via an optimization over perturbations, the rate functional obtained turns out to be expressed by a variational expression. On the other hand, for the large deviations lower bound, it is required to find the cheapest perturbation that leads the system to a given profile distinct from the expected limit. In other words, it is necessary to solve the variational expression of the rate function, at least for a sufficiently large class of density profiles. This is precisely what we do in the large deviations lower bound, by means of a proof surprisingly simple. In fact, the proof (of Proposition 6.1) consists essentially in checking that the perturbation H that leads the system to a limit ρ^H is the cheapest one. Indeed, a difficult part of the work was to find the correct class of perturbations for the dynamics and fulfill the technical details.

Then, since the rate functional is convex in a specific sense, by a density argument we extend the lower bound for the class of smooth profiles. The extension for general profiles is a hard problem of convex analysis and illustrates that there is much to be developed in terms in of Orlicz spaces as devices in large deviations schemes. This is subject of future work.

The paper is organized as follows. In Section 2, we introduce notation and state the main results, namely Theorem 2.8 and Theorem 2.12. In Section 3, we establish the replacement lemma and the energy estimates. In Section 4, we prove

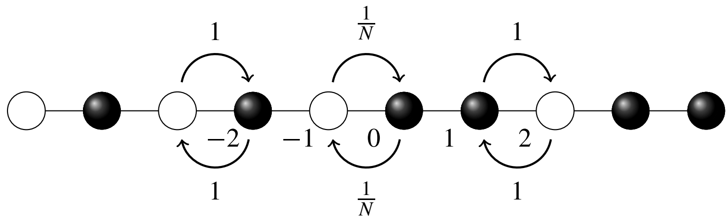

FIG. 1. *The bond of vertices* $\{-1, 0\}$, *the slow bond, has particular rates associated to it.*

Theorem 2.8. In Section 5, we prove the upper bound. Finally, the lower bound for smooth profiles is presented in the Section 6.

2. Model and statements. Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z} = \{0, 1, 2, \dots, N-1\}$ be the one-dimensional discrete torus with N points. In each site of $\mathbb{T}_N$, we allow at most one particle. In other words, we consider configurations of particles $\eta \in \{0, 1\}^{\mathbb{T}_N}$. We say that $\eta(x) = 0$ if the site $x \in \mathbb{T}_N$ is vacant, and $\eta(x) = 1$ if the site $x \in \mathbb{T}_N$ is occupied. Notice that $x = 0$ and $x = N$ are the same site. Denote by $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$ this state space.

The exclusion process with a slow bond at the bond of vertices $-1, 0$, which has been considered in [4, 6, 7], can be described as follows. To each bond of $\mathbb{T}_N$, we associate a Poisson clock, and these are assumed to be independent. If the bond is at the vertices $-1, 0$, the parameter of the Poisson is taken as $1/N$. All the other Poisson clocks have parameter one. When a clock rings, the occupation values of η at the vertices of the associated bond are exchanged. The smaller parameter at the bond of vertices $-1, 0$ slows the passage of particles crossing it, hence we get the name *slow bond* (see Figure 1).

This Markov process can also be characterized in terms of its infinitesimal generator L_N, which acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(L_N f)(\eta) = \frac{1}{N}[f(\eta^{-1,0}) - f(\eta)] + \sum_{\substack{x \in \mathbb{T}_N \\ x \neq -1}} [f(\eta^{x,x+1}) - f(\eta)], \tag{2.1}$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x), \eta(x+1)$: $\eta^{x,x+1}(x) = \eta(x+1)$, $\eta^{x,x+1}(x+1) = \eta(x)$ and $\eta^{x,x+1}(y) = \eta(y)$, if $y \neq x, x+1$.

Denote by $\{\eta_t; t \geq 0\}$ the Markov process on $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$ associated with the generator L_N, defined in (2.1), *speeded up* by N^2. The dependency of η_t in N will be omitted to keep notation as simple as possible.

Throughout the paper, we fix a time-horizon $T > 0$. Consider the trajectories of the Markov process η_t with $t \in [0, T]$. Let $\mathcal{D}([0, T], \Omega_N)$ be the path space of càdlàg time trajectories taking values on $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$. For short, we will denote this space just by $\mathcal{D}_{\Omega_N}$. Given a measure μ_N on Ω_N, denote by $\mathbb{P}_{\mu_N}$ the probability measure on $\mathcal{D}_{\Omega_N}$ induced by the initial state μ_N and the Markov process $\{\eta_t; t \geq 0\}$. Expectation with respect to $\mathbb{P}_{\mu_N}$ will be denoted by $\mathbb{E}_{\mu_N}$. Let ν_α^N be the Bernoulli product measure on Ω_N with marginals given by

$$\nu_\alpha^N\{\eta; \eta(x) = 1\} = \alpha \qquad \forall x \in \mathbb{T}_N.$$

These measures $\{\nu_\alpha^N; 0 \leq \alpha \leq 1\}$ are invariant, in fact reversible, for the dynamics described above. Denote by $\mathbb{T} = [0, 1]$ the one-dimensional continuous torus, where we identify the points 0 and 1.

DEFINITION 2.1. A sequence of probability measures $\{\mu_N; N \geq 1\}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0, 1]$ if

$$\lim_{N\to\infty} \mu_N\left[\eta; \left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\left(\frac{x}{N}\right)\eta(x) - \int H(u)\rho_0(u)\,du\right| > \delta\right] = 0, \tag{2.2}$$

for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \to \mathbb{R}$.

The quantity introduced in the definition above can be reformulated in terms of empirical measures. We start by defining the set

$$\mathcal{M} = \{\mu; \mu \text{ is a positive measure on } \mathbb{T} \text{ with } \mu(\mathbb{T}) \leq 1\}, \tag{2.3}$$

this space is endowed with the weak topology. Consider the measure $\pi^N \in \mathcal{M}$, which is obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi^N(\eta, du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta(x)\delta_{\frac{x}{N}}(du),$$

where δ_u is the Dirac measure concentrated on u. The measure $\pi^N(\eta, du)$ is called the empirical measure associated to the configuration η. With this notation, $\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(\frac{x}{N})\eta(x)$ is the integral of H with respect to the empirical measure π^N, denoted by $\langle\pi^N, H\rangle$.

We consider the time evolution of the empirical measure π_t^N associated to the Markov process $\{\eta_t; t \geq 0\}$ by

$$\pi_t^N(du) = \pi^N(\eta_t, du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\delta_{\frac{x}{N}}(du). \tag{2.4}$$

Note that (2.2) is equivalent to saying that π_0^N converges in distribution to $\rho_0(u)\,du$. Let $\mathcal{D}([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0, T] \to \mathcal{M}$ endowed with the *Skorohod* topology. For short, we will use the notation $\mathcal{D}_\mathcal{M} = \mathcal{D}([0, T], \mathcal{M})$. Denote by $\mathbb{Q}_{\mu_N}^N$ the measure on the path space $\mathcal{D}_\mathcal{M}$ induced by the measure μ_N and the empirical process π_t^N introduced in (2.4).

2.1. *Frequently used notation.* Before stating results we present some important notation to be used throughout paper.

- The indicator function of a set A will be written by $\mathbf{1}_A(u)$, which is one when $u \in A$ and zero otherwise.
- Given a function $H : \mathbb{T} \to \mathbb{R}$, we will denote $H(0^-)$ and $H(0^+)$, respectively, for the left and right side limits of H at the point $0 \in \mathbb{T}$.
- Given a function $H : \mathbb{T} \to \mathbb{R}$, denote $\delta H(0) = H(0^+) - H(0^-)$ its jump size at zero. And denote $\delta_N H_x = H(\frac{x+1}{N}) - H(\frac{x}{N})$. Hence, provided H is right continuous at zero, $\delta_N H_{-1}$ converges to $\delta H(0)$.
- Given a function $g : [0, T] \times \mathbb{T}$, we write $g_t(u)$ to denote $g(t, u)$. It should not be misunderstood with the notation for time derivative, namely $\partial_t g(t, u)$.
- Given a nonnegative integer k, denote by $C^k(\mathbb{T})$ the set of real-valued functions with domain $\mathbb{T}$ with continuous derivatives up to order k. As natural, $C(\mathbb{T})$ denotes the set of continuous functions. For nonnegative integers j and k, denote by $C^{j,k}([0, T] \times \mathbb{T})$ the set of real valued functions with domain $[0, T] \times \mathbb{T}$ with continuous derivatives up to order j in the first variable (time), and continuous derivatives up to order k in the second variable (space).
- The notation C_{k} means compact support contained in $[0, T] \times (0, 1)$. For instance, $C_{\mathrm{k}}^{j,k}([0, T] \times (0, 1))$ denotes the subset of $C^{j,k}([0, T] \times (0, 1))$ composed of functions with compact support contained in $[0, T] \times (0, 1)$.
- The notation $g(N) = O(f(N))$ means $g(N)$ is bounded from above by $Cf(N)$, where the constant C does not depend on N. The notation $g(N) = o(f(N))$ means $\lim_{N\to\infty} g(N)/f(N) = 0$.
- Despite we have denoted $\langle \pi_t^N, H\rangle = \frac{1}{N}\sum_{x\in\mathbb{T}_N} H(\frac{x}{N})\eta_t(x)$, the bracket $\langle\cdot,\cdot\rangle$ will also mean the inner product in $L^2(\mathbb{T})$ and in $L^2[0, 1]$. The double bracket $\langle\langle\cdot,\cdot\rangle\rangle$ will denote the inner product in $L^2([0, T] \times \mathbb{T})$.

2.2. *The hydrodynamic equation.* The slow bond, as we will see, yields a discontinuity at the origin in the continuum limit. Therefore, discontinuous functions at the origin are naturally required.

DEFINITION 2.2. Denote by $C^{1,2}([0, T] \times [0, 1])$ the space of functions $H : [0, T] \times \mathbb{T} \to \mathbb{R}$ such that:

1. H restricted to $[0, T] \times \mathbb{T}\backslash\{0\}$ belongs to $C^{1,2}([0, T] \times \mathbb{T}\backslash\{0\})$.
2. Identifying $\mathbb{T}\backslash\{0\}$ with the open interval $(0, 1)$, H has a $C^{1,2}$ extension to $[0, T] \times [0, 1]$.
3. For any $t \in [0, T]$, H is right continuous at zero, that is, $H(t, 0) = \lim_{x\to 0^+} H(t, x)$.

This space of test functions should not be misunderstood with $C^{1,2}([0, T] \times \mathbb{T})$. In words, a function H belongs to this space $C^{1,2}([0, T] \times [0, 1])$ if, "opening" the torus at 0, the function has a $C^{1,2}$ extension to the closed interval $[0, 1]$.

DEFINITION 2.3 (Sobolev space). Let $\mathcal{H}^1(0,1)$ be the set of all locally summable functions $\zeta : (0,1) \to \mathbb{R}$ such that there exists a function $\partial_u \zeta \in L^2(0,1)$ satisfying $\langle \partial_u G, \zeta \rangle = -\langle G, \partial_u \zeta \rangle$, for all $G \in C_{\mathrm{k}}^{\infty}((0,1))$. For $\zeta \in \mathcal{H}^1(0,1)$, we define the norm

$$\|\zeta\|_{\mathcal{H}^1(0,1)} := \left(\|\zeta\|^2_{L^2(0,1)} + \|\partial_u \zeta\|^2_{L^2(0,1)}\right)^{1/2}.$$

Let $L^2(0,T;\mathcal{H}^1(0,1))$ be the space of all measurable functions $\xi : [0,T] \to \mathcal{H}^1(0,1)$ such that

$$\|\xi\|^2_{L^2(0,T;\mathcal{H}^1(0,1))} := \int_0^T \|\xi_t\|^2_{\mathcal{H}^1(0,1)}\, dt < \infty.$$

REMARK 2.4. An equivalent and useful definition for the Sobolev space $L^2(0,T;\mathcal{H}^1(0,1))$ is the set of bounded functions $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that there exists a function $\partial \xi \in L^2([0,T] \times \mathbb{T})$ satisfying

$$\langle\langle \partial_u H, \xi \rangle\rangle = -\langle\langle H, \partial \xi \rangle\rangle,$$

for all functions $H \in C_{\mathrm{k}}^{0,1}([0,T] \times (0,1))$.

DEFINITION 2.5 (The hydrodynamic equation for the SSEP with a slow bond). Consider a measurable density profile $\gamma : \mathbb{T} \to [0,1]$. A function $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is said to be a weak solution of the parabolic differential equation with Robin boundary conditions

$$\begin{cases} \partial_t \rho = \Delta \rho, \\ \rho_0(\cdot) = \gamma(\cdot), \\ \partial_u \rho_t(0^+) = \partial_u \rho_t(0^-) = \rho_t(0^+) - \rho_t(0^-), \end{cases} \tag{2.5}$$

if the following two conditions are fulfilled:

(1) $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$.
(2) For all functions $G \in C^{1,2}([0,T] \times [0,1])$ and for all $t \in [0,T]$, ρ satisfies the integral equation

$$\begin{aligned} \langle \rho_t, G_t \rangle - \langle \gamma, G_0 \rangle = {} & \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle\, ds \\ & + \int_0^t \{\rho_s(0^+)\partial_u G_s(0^+) - \rho_s(0^-)\partial_u G_s(0^-)\}\, ds \\ & - \int_0^t (\rho_s(0^+) - \rho_s(0^-))(G_s(0^+) - G_s(0^-))\, ds. \end{aligned} \tag{2.6}$$

Assumption (1) guarantees that the boundary integrals are well-defined. The Robin (mixed) boundary conditions in (2.5) can be interpreted as the Fick law at the point $x = 0$. This is discussed in more detail in [4]. The uniqueness and existence of weak solutions of (2.5) was proved in [6]. Moreover, it was proved in [4, 6, 7] and we have the following.

THEOREM 2.6. *Fix a measurable density profile $\gamma : \mathbb{T} \to [0,1]$ and consider a sequence of probability measures μ_N on Ω_N associated to γ in the sense of* (2.2). *Then, for any $t \in [0,T]$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\eta_t(x) - \int G(u)\rho_t(u)\,du\right| > \delta\right] = 0, \tag{2.7}$$

for every $\delta > 0$ and every function $G \in C(\mathbb{T})$. Here, ρ is the unique weak solution of the linear partial differential equation (2.5) *with $\rho_0 = \gamma$.*

2.3. *The weakly asymmetric exclusion process with a slow bond.* In order to obtain the large deviations of a Markov process, a natural step is to prove the LLN for a class of perturbations of the original Markov process. In our case, the correct perturbations will be given by the class of *weakly asymmetric exclusion processes with a slow bond*, to be defined ahead. For short, we will call it just *WASEP with a slow bond*.

Recall Definition 2.2. Given a function $H \in C^{1,2}([0,T]\times[0,1])$, consider the time nonhomogeneous Markov process whose generator at time t acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$\begin{aligned}&\big(L_{N,t}^H f\big)(\eta)\\ &\quad= \sum_{x\in\mathbb{T}_N} \xi_x^N e^{H_t(\frac{x+1}{N})-H_t(\frac{x}{N})}\eta(x)\big(1-\eta(x+1)\big)\big[f\big(\eta^{x,x+1}\big)-f(\eta)\big]\\ &\qquad+\sum_{x\in\mathbb{T}_N} \xi_x^N e^{-H_t(\frac{x+1}{N})+H_t(\frac{x}{N})}\eta(x+1)\big(1-\eta(x)\big)\big[f\big(\eta^{x,x+1}\big)-f(\eta)\big],\end{aligned} \tag{2.8}$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$, $\eta(x+1)$, and

$$\xi_x^N = \begin{cases} 1 & \text{if } x \in \mathbb{T}_N\backslash\{-1\},\\ N^{-1} & \text{if } x = -1.\end{cases} \tag{2.9}$$

In the particular case H is a constant function, the generator $L_{N,t}^H$ turns out to be equal to the generator L_N defined in (2.1). We emphasize that the asymmetry is weak in all the bonds except at the bond of vertices $-1, 0$. Since the function H is possibly discontinuous at the origin, the asymmetry in that bond does not go to zero in the limit, appearing indeed in the hydrodynamical equation.

Let $\{\eta_t^H; t \geq 0\}$ be the nonhomogeneous Markov process with generator $L_{N,t}^H$ defined in (2.8) *speeded up by N^2*. Given a probability measure μ_N on Ω_N, denote by $\mathbb{P}_{\mu_N}^H$ the probability measure on the space of trajectories $\mathcal{D}_{\Omega_N}$ induced by the Markov process $\{\eta_t^H; t \geq 0\}$ starting from the measure μ_N.

DEFINITION 2.7 (Hydrodynamic equation for the WASEP with a slow bond). Let $\gamma : \mathbb{T} \to \mathbb{R}$ be a bounded density profile and fix $H \in C^{1,2}([0,T] \times [0,1])$. A function $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is said to be a weak solution of the partial differential equation

$$\begin{cases} \partial_t \rho = \Delta\rho - 2\partial_u\big(\chi(\rho)\partial_u H\big), \\ \rho_0(\cdot) = \gamma(\cdot), \\ \partial_u \rho_t(0^+) = 2\chi(\rho_t(0^+))\partial_u H_t(0^+) - \varphi_t(\rho, H), \\ \partial_u \rho_t(0^-) = 2\chi(\rho_t(0^-))\partial_u H_t(0^-) - \varphi_t(\rho, H), \end{cases} \tag{2.10}$$

where $\chi(\alpha) = \alpha(1-\alpha)$ and

$$\varphi_t(\rho, H) = \rho_t(0^-)(1-\rho_t(0^+))e^{\delta H_t(0)} - \rho_t(0^+)(1-\rho_t(0^-))e^{-\delta H_t(0)}, \tag{2.11}$$

if the following two conditions are fulfilled:

(1) $\rho \in L^2(0,T; \mathcal{H}^1(0,1))$.

(2) For all functions G in $C^{1,2}([0,T] \times [0,1])$, and all $t \in [0,T]$, ρ satisfies the integral equation

$$\begin{aligned} \langle \rho_t, G_t\rangle - \langle \gamma, G_0\rangle = {} & \int_0^t \langle \rho_s, (\partial_s + \Delta)G_s\rangle\, ds + 2\int_0^t \langle \chi(\rho_s)\partial_u H_s, \partial_u G_s\rangle\, ds \\ & + \int_0^t \{\rho_s(0^+)\partial_u G_s(0^+) - \rho_s(0^-)\partial_u G_s(0^-)\}\, ds \\ & + \int_0^t \varphi_s(\rho, H)\delta G_s(0)\, ds. \end{aligned} \tag{2.12}$$

The nonlinearity in mixed boundary conditions of (2.10) lead to a very complicated problem of uniqueness. Existence of weak solutions of (2.10) is a consequence of the tightness of the process, as we will see in Section 4. The assumption on uniqueness of weak solutions of (2.10) is also needed in the proof of large deviations, because its proof depends on the hydrodynamic limit for the WASEP with a slow bond.

Our first result is the hydrodynamic limit for the WASEP with a slow bond.

THEOREM 2.8. *Suppose uniqueness of weak solutions of PDE* (2.10). *Let* $H \in C^{1,2}([0,T] \times [0,1])$. *Fix a continuous initial profile* $\gamma : \mathbb{T} \to [0,1]$ *and consider a sequence of probability measures* μ_N *on* $\{0,1\}^{\mathbb{T}_N}$ *associated to* γ *in the sense* (2.2). *Then, for any* $t \in [0,T]$,

$$\lim_{N\to\infty} \mathbb{P}^H_{\mu_N}\bigg[\bigg|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\Big(\frac{x}{N}\Big)\eta^H_t(x) - \int G(u)\rho_t(u)\, du\bigg| > \delta\bigg] = 0,$$

for every $\delta > 0$ *and every function* $G \in C(\mathbb{T})$, *where* ρ *is the unique weak solution of* (2.10) *with* $\rho_0 = \gamma$.

2.4. *Large deviations principle.* Denote by $\mathcal{M}_0$ the subset of $\mathcal{M}$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_0 = \{\omega \in \mathcal{M}; \omega(du) = \rho(u)\, du \text{ and } 0 \leq \rho \leq 1 \text{ almost surely}\}.$$

The set $\mathcal{M}_0$ is a closed subset of $\mathcal{M}$ endowed with the weak topology. This property is inherited by $\mathcal{D}([0,T], \mathcal{M}_0)$, which is a closed subset of $\mathcal{D}_{\mathcal{M}}$ for the Skorohod topology. We will denote $\mathcal{D}([0,T], \mathcal{M}_0)$ simply by $\mathcal{D}_{\mathcal{M}_0}$.

DEFINITION 2.9. Given $H \in C_{\mathrm{k}}^{0,1}([0,T] \times (0,1))$ define $\mathcal{E}_H : \mathcal{D}_{\mathcal{M}} \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_H(\pi) = \begin{cases} \langle\langle \partial_u H, \rho \rangle\rangle - 2\langle\langle H, H \rangle\rangle & \text{if } \pi \in \mathcal{D}_{\mathcal{M}_0} \text{ and } \pi(du) = \rho(t,u)\, du, \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, define the energy functional $\mathcal{E} : \mathcal{D}_{\mathcal{M}} \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\mathcal{E}(\pi) = \sup_H \mathcal{E}_H(\pi), \tag{2.13}$$

where the supremum is taken over functions $H \in C_{\mathrm{k}}^{0,1}([0,T] \times (0,1))$.

In Section 3.5 we prove that if $\pi \in \mathcal{D}_{\mathcal{M}}$ and $\mathcal{E}(\pi) < \infty$, then there exists $\rho \in L^2(0,T; \mathcal{H}^1(0,1))$ such that $\pi(t,du) = \rho_t(u)\, du$. Keeping this in mind, given $H \in C^{1,2}([0,T] \times [0,1])$ and $\pi \in \mathcal{D}_{\mathcal{M}}$, define

$$\hat{J}_H(\pi) = \ell_H(\pi) - \Phi_H(\pi), \tag{2.14}$$

where

$$\begin{aligned} \ell_H(\pi) = \langle \rho_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T \langle \rho_t, (\partial_t + \Delta) H_t \rangle\, dt \\ - \int_0^T \{\rho_t(0^+)\partial_u H_t(0^+) - \rho_t(0^-)\partial_u H_t(0^-)\}\, dt \\ + \int_0^T (\rho_t(0^+) - \rho_t(0^-))\delta H_t(0)\, dt \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \Phi_H(\pi) = \int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2 \rangle\, dt + \int_0^T \rho_t(0^-)(1-\rho_t(0^+))\psi(\delta H_t(0))\, dt \\ + \int_0^T \rho_t(0^+)(1-\rho_t(0^-))\psi(-\delta H_t(0))\, dt, \end{aligned} \tag{2.16}$$

where $\psi(x) = e^x - x - 1$ and $\delta H_t(0) = H_t(0^+) - H_t(0^-)$. It is worth highlighting that, as functions of H, $\ell_H(\pi)$ is linear and $\Phi_H(\pi)$ is convex.

DEFINITION 2.10. Given $H \in C^{1,2}([0,T]\times[0,1])$, define the functional $J_H : \mathcal{D}_{\mathcal{M}} \to \mathbb{R}\cup\{\infty\}$ by

$$J_H(\pi) = \begin{cases} \hat{J}_H(\pi) & \text{if } \mathcal{E}(\pi) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

DEFINITION 2.11. Let the rate functional $I : \mathcal{D}_{\mathcal{M}} \to \mathbb{R}_+ \cup \{\infty\}$ be

$$I(\pi) = \sup_H J_H(\pi),$$

being the supremum above over functions $H \in C^{1,2}([0,T]\times[0,1])$.

The large deviations study is decomposed into the study of deviations from the initial measure and that of deviations from the expected trajectory; see [9], Chapter 10. Since the large deviations for Bernoulli product measures are well known, we restrict ourselves to the deviations from the expected trajectory. We start henceforth the process from a sequence of *deterministic* initial configurations. This avoids the analysis of statical large deviations, since we interested here in dynamical large deviations. Recall that $\mathbb{Q}^N_{\mu_N}$ is the measure on the path space $\mathcal{D}_{\mathcal{M}}$ induced by the initial measure μ_N and the empirical process π^N_t introduced in (2.4). We are now in position to state the main result of the paper.

THEOREM 2.12. *Let μ_N be a sequence of deterministic initial configurations associated to a bounded density profile $\gamma : \mathbb{T} \to \mathbb{R}$ in the sense of the Definition* 2.1. *Then the sequence of measures $\{\mathbb{Q}^N_{\mu_N}; N \geq 1\}$ satisfies the following large deviation estimates*:

(i) Upper bound: *For any $\mathcal{C}$ closed subset of $\mathcal{D}_{\mathcal{M}}$*,

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}^N_{\mu_N}[\mathcal{C}] \leq -\inf_{\pi\in\mathcal{C}} I(\pi).$$

(ii) Lower bound for smooth profiles: *For any $\mathcal{O}$ open subset of $\mathcal{D}_{\mathcal{M}}$*,

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{Q}^N_{\mu_N}[\mathcal{O}] \geq -\inf_{\pi\in\mathcal{O}\cap\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}} I(\pi),$$

where $\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}$ denotes the set of paths $\pi \in \mathcal{D}_{\mathcal{M}}$ such that $\pi_t(du) = \rho_t(u)\,du$ with $\rho \in C^{1,2}([0,T]\times[0,1])$.

Item (i) of Theorem 2.12 is proved in Section 5. The proof of item (ii) is presented in Section 6.

3. Super-exponential replacement lemmas and energy estimate. Both in the proof of hydrodynamic limit for the WASEP with a slow bond and in the proof of the large deviations principle for the SSEP with a slow bond, replacement lemma and energy estimates play an important role. By a *replacement lemma*, we mean a result that allows to replace the average time occupation in a site for the average time occupation in a box around that site. And by *energy estimates* we mean a result assuring that time trajectories of the empirical measure are asymptotically close to elements of a certain Sobolev space. In the proof of large deviations, we will need such results in a super-exponential setting. In other words, the corresponding probabilities must converge to one in a faster way than exponentially.

Proofs omitted in this section can be found in the extended version [8].

3.1. *Definitions and estimates lemmas.* Denote by $\mathbf{H}(\mu_N|\nu_\alpha^N)$ the entropy of a probability measure μ_N with respect to the invariant measure ν_α^N. For a precise definition and properties of the entropy, see [9]. It is well known the existence of a constant $K_0 := K_0(\alpha)$ such that

$$\mathbf{H}(\mu_N|\nu_\alpha^N) \leq K_0 N, \tag{3.1}$$

for any probability measure μ_N in Ω_N. See, for instance, the Appendix of [4]. Denote by $\mathfrak{D}_N$ the Dirichlet form, which is defined by

$$\mathfrak{D}_N(f) = \langle -L_N\sqrt{f}, \sqrt{f}\rangle_{\nu_\alpha^N},$$

where f is a probability density with respect to ν_α^N. An elementary computation shows that

$$\mathfrak{D}_N(f) = \sum_{x\in\mathbb{T}_N} \frac{\xi_x^N}{2} \int \big(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\big)^2 \, d\nu_\alpha^N(\eta),$$

where ξ_x^N is defined in (2.9).

From this point on, abusing notation, we denote the greatest integer less than or equal to εN simply by εN. Next, we define the local average of particles, which corresponds to the mean occupation in a box around a given site. The idea is to define a box around the site x in such a way it avoids the slow bond.

DEFINITION 3.1. If $x \in \mathbb{T}_N$ is such that $\frac{x}{N} \in \mathbb{T}\backslash(-\varepsilon, 0)$, we define the local average by

$$\eta^{\varepsilon N}(x) = \frac{1}{\varepsilon N} \sum_{y=x+1}^{x+\varepsilon N} \eta(y).$$

If $\frac{x}{N} \in (-\varepsilon, 0)$, define the local average by

$$\eta^{\varepsilon N}(x) = \frac{1}{\varepsilon N} \sum_{y=-\varepsilon N}^{-1} \eta(y).$$

In accordance with to the previous definition of local density of particles, we define an approximation of identity ι_ε in the continuous torus by

$$\iota_\varepsilon(u,v) = \begin{cases} \frac{1}{\varepsilon}\mathbf{1}_{(v,v+\varepsilon)}(u) & \text{if } v \in \mathbb{T}\backslash(-\varepsilon,0), \\ \frac{1}{\varepsilon}\mathbf{1}_{(-\varepsilon,0)}(u) & \text{if } v \in (-\varepsilon,0). \end{cases} \tag{3.2}$$

We also define the convolution $(\psi * \iota_\varepsilon)(v) = \langle \psi, \iota_\varepsilon(\cdot, v)\rangle$, for a function $\psi : \mathbb{T} \to \mathbb{R}$ or a measure ψ on the torus $\mathbb{T}$. The following identity is relevant:

$$(\pi^N * \iota_\varepsilon)\Big(\frac{x}{N}\Big) = \eta^{\varepsilon N}(x) \qquad \text{for all } x \in \mathbb{T}_N. \tag{3.3}$$

To simplify notation, define the functions

$$g_1 : \{0,1\}^{\mathbb{T}} \to \mathbb{R} \qquad \text{by } g_1(\eta) = \eta(0)\big(1-\eta(1)\big) \tag{3.4}$$

and

$$\tilde{g}_1 : [0,1]\times[0,1] \to \mathbb{R} \qquad \text{by } \tilde{g}_1(\alpha,\beta) = \alpha(1-\beta).$$

Also,

$$g_2 : \{0,1\}^{\mathbb{T}} \to \mathbb{R} \qquad \text{by } g_2(\eta) = \eta(1)\big(1-\eta(0)\big) \tag{3.5}$$

and

$$\tilde{g}_2 : [0,1]\times[0,1] \to \mathbb{R} \qquad \text{by } \tilde{g}_2(\alpha,\beta) = \beta(1-\alpha).$$

LEMMA 3.2. *Fix a function $F : \mathbb{T} \to \mathbb{R}$ and let f be a density with respect to ν_α^N. Then, for any $A > 0$, the following inequalities hold*:

$$\begin{aligned} &\frac{1}{N}\sum_{x\neq -1}\int F\Big(\frac{x}{N}\Big)\big\{\tau_x g_i(\eta) - \tilde{g}_i\big(\eta^{\varepsilon N}(x), \eta^{\varepsilon N}(x+1)\big)\big\} f(\eta)\, d\nu_\alpha^N(\eta) \\ &\quad \le 12A\varepsilon \sum_{x\neq -1}\Big(F\Big(\frac{x}{N}\Big)\Big)^2 + \frac{3}{A}\mathfrak{D}_N(f), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\frac{1}{N}\sum_{x\in\mathbb{T}_N}\int F\Big(\frac{x}{N}\Big)\big\{\eta(x) - \eta^{\varepsilon N}(x)\big\} f(\eta)\, d\nu_\alpha^N(\eta) \\ &\quad \le 4A\varepsilon \sum_{x\in\mathbb{T}_N}\Big(F\Big(\frac{x}{N}\Big)\Big)^2 + \frac{1}{A}\mathfrak{D}_N(f), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &F\Big(\frac{-1}{N}\Big)\int \big\{\tau_{-1} g_i(\eta) - \tilde{g}_i\big(\eta^{\varepsilon N}(-1), \eta^{\varepsilon N}(0)\big)\big\} f(\eta)\, d\nu_\alpha^N(\eta) \\ &\quad \le 6A\varepsilon N\Big(F\Big(\frac{-1}{N}\Big)\Big)^2 + \frac{3}{A}\mathfrak{D}_N(f), \end{aligned} \tag{3.8}$$

$$\int \big\{\eta(x) - \eta^{\varepsilon N}(x)\big\} f(\eta)\, d\nu_\alpha^N(\eta) \le 4NA\varepsilon + \frac{1}{A}\mathfrak{D}_N(f), \qquad \forall x \in \mathbb{T}_N, \tag{3.9}$$

with $i = 1, 2$.

LEMMA 3.3. *Fix any function $H : \mathbb{T} \to \mathbb{R}$ and let f be a density with respect to ν_α^N. Then*

$$\begin{aligned}&\int \frac{1}{\varepsilon N} \sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\{\eta(x-\varepsilon N)-\eta(x)\} f(\eta)\, d\nu_\alpha^N(\eta)\\ &\qquad \le N\mathfrak{D}_N(f) + \frac{2}{N}\sum_{x\in\mathbb{T}_N}\Big(H\Big(\frac{x}{N}\Big)\Big)^2\Big\{1+\frac{1}{\varepsilon}\mathbf{1}_{(-\varepsilon,0]}\Big(\frac{x}{N}\Big)\Big\}.\end{aligned} \tag{3.10}$$

Moreover, this inequality remains valid replacing $\{\eta(x-\varepsilon N)-\eta(x)\}$ by $\{\eta(x)-\eta(x+\varepsilon N)\}$.

3.2. *Super-exponential replacement lemmas.* In the large deviations proof, the replacement lemma presented in Section 3.9 is not enough because we need to prove that the difference between cylinder functions and functions of the density field are super-exponentially small, that is, of order smaller that $\exp\{-CN\}$, for any $C > 0$. We begin by exhibiting a super-exponential replacement for the invariant measure ν_α^N.

PROPOSITION 3.4. *Let $F_i : [0,T]\times\mathbb{T} \to \mathbb{R}$, $i = 1, 2$, such that*

$$\varlimsup_{N\to\infty}\int_0^T\Big(\Big(F_2\Big(t,\frac{-1}{N}\Big)\Big)^2 + \frac{1}{N}\sum_{x\neq -1}\Big(F_1\Big(t,\frac{x}{N}\Big)\Big)^2\Big)\,dt < \infty.$$

For each $\varepsilon > 0$, consider

$$\begin{aligned}V_{N,\varepsilon}^{F_1,F_2}(t,\eta) := {}& \frac{1}{N}\sum_{x\neq -1} F_1\Big(t,\frac{x}{N}\Big)\{\tau_x g_1(\eta) - \tilde g_1(\eta^{\varepsilon N}(x),\eta^{\varepsilon N}(x+1))\}\\ &+ F_2\Big(t,\frac{-1}{N}\Big)\{\tau_{-1}g_1(\eta)-\tilde g_1(\eta^{\varepsilon N}(-1),\eta^{\varepsilon N}(0))\},\end{aligned}$$

where g_1 and $\tilde g_1$ have been defined in (3.4). Then, for any $\delta > 0$,

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log \mathbb{P}_{\nu_\alpha^N}\Big[\Big|\int_0^T V_{N,\varepsilon}^{F_1,F_2}(t,\eta_t)\,dt\Big| > \delta\Big] = -\infty. \tag{3.11}$$

Finally, it is true the same result with g_2 and $\tilde g_2$ in lieu of g_1 and $\tilde g_1$.

COROLLARY 3.5. *Under the same hypothesis of the Proposition 3.4, for any $\delta > 0$,*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log \mathbb{P}_{\mu_N}\Big[\Big|\int_0^T V_{N,\varepsilon}^{F_1,F_2}(t,\eta_t)\,dt\Big| > \delta\Big] = -\infty. \tag{3.12}$$

Finally, the same result is still valid with g_2 and $\tilde g_2$ in lieu of g_1 and $\tilde g_1$.

COROLLARY 3.6. *Given a bounded function* $F : [0, T] \times \mathbb{T}$ *and* $x = -1$ *or* $x = 0$, *let*

$$\hat{V}^{F,x}_{N,\varepsilon}(t, \eta) = F\left(t, \frac{x}{N}\right)\{\eta(x) - \eta^{\varepsilon N}(x)\}.$$

Then, for any $\delta > 0$,

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}\left[\left|\int_0^T \hat{V}^{F,x}_{N,\varepsilon}(t, \eta_t)\, dt\right| > \delta\right] = -\infty. \tag{3.13}$$

3.3. *Super-exponential energy estimate.* Our goal here is to exclude trajectories with infinite energy in the large deviations regime. The next proposition is the key in the energy estimates.

PROPOSITION 3.7. *For any function* $H \in C^{0,1}_{\mathrm{k}}([0, T] \times (0, 1))$,*and* E_H *defined in Definition* 2.9, *the following inequality holds*:

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\mathcal{E}_H(\pi^N * \iota_\varepsilon) \ge l] \le -l + K_0, \qquad \forall l \in \mathbb{R}.$$

COROLLARY 3.8. *For any functions* $H_1, \dots, H_k \in C^{0,1}_{\mathrm{k}}([0, T] \times (0, 1))$ *holds*

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k} \mathcal{E}_{H_j}(\pi^N * \iota_\varepsilon) \ge l\Big] \le -l + K_0. \tag{3.14}$$

3.4. *Replacement lemma.* The result of this subsection is a consequence of the super-exponential replacement lemmas and it is used in the proof of hydrodynamic limit.

PROPOSITION 3.9 (Replacement lemma). *Let* $F : \mathbb{T} \to \mathbb{R}$ *be a bounded function and* $(\mu_N)_{N\ge 1}$ *any sequence of measures. Then, for all* $i = 1, 2$ *and* $t \in [0, T]$, *we have*

$$\begin{aligned}
&\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \frac{1}{N} \sum_{x\ne -1} F\left(\frac{x}{N}\right)\{\tau_x g_i(\eta_s)\right.\right. \\
&\qquad \left.\left. - \tilde{g}_i\big(\eta^{\varepsilon N}_s(x), \eta^{\varepsilon N}_s(x+1)\big)\}\, ds\right|\right] = 0,
\end{aligned}$$

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} F\left(\frac{x}{N}\right)\{\eta_s(x) - \eta^{\varepsilon N}_s(x)\}\, ds\right|\right] = 0, \tag{3.15}$$

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \{\tau_{-1} g_i(\eta_s) - \tilde{g}_i\big(\eta^{\varepsilon N}_s(-1), \eta^{\varepsilon N}_s(0)\big)\}\, ds\right|\right] = 0,$$

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^T \{\eta_s(x) - \eta^{\varepsilon N}_s(x)\}\, ds\right|\right] = 0, \qquad \textit{for } x = -1, 0.$$

3.5. *Sobolev space*. We prove in this section that any limit point $\mathbb{Q}^*$ of the sequence $\mathbb{Q}^N_{\mu_N}$ is concentrated on trajectories $\rho(t,u)\,du$, where ρ belongs to a certain Sobolev space to be defined ahead. Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^N_{\mu_N}$ along some subsequence.

PROPOSITION 3.10. *The measure $\mathbb{Q}^*$ is concentrated on paths $\rho_t(u)\,du$ such that $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$.*

The proof is based on the Riesz representation theorem and follows from the next lemma.

LEMMA 3.11.

$$E_{\mathbb{Q}^*}\left[\sup_H\left\{\int_0^T\int_{\mathbb{T}}\partial_u H(s,u)\rho_s(u)\,du\,ds - 2\int_0^T\int_{\mathbb{T}}H(s,u)^2\,du\,ds\right\}\right] \le K_0,$$

where the supremum is carried over all functions H in $C^{0,1}_{\mathrm{k}}([0,T]\times(0,1))$.

4. Hydrodynamic limit of the WASEP with a slow bond. The empirical measure π^N_t corresponding to $\{\eta^H_t; t \ge 0\}$ is defined in the same way of (2.4). Denote by $\mathbb{Q}^H_{\mu_N}$ the probability measure on the space of trajectories $\mathcal{D}_{\mathcal{M}}$ induced by the empirical measure π^N_t.

PROPOSITION 4.1. *Consider a bounded density profile $\rho_0 : \mathbb{T} \to \mathbb{R}$ and $H \in C^{1,2}([0,T]\times[0,1])$. The sequence of probabilities $\{\mathbb{Q}^H_{\mu_N}; N \ge 1\}$ converges in distribution to the probability measure concentrated on the absolutely continuous path $\pi_t(du) = \rho_t(u)\,du$, where density ρ is the unique weak solution of the partial differential equation* (2.10).

Observe that Theorem 2.8 is a corollary of the previous proposition. The proof of above is divided in two parts. In Section 4.1, we show that the sequence $\{\mathbb{Q}^H_{\mu_N}; N \ge 1\}$ is tight. Section 4.4 is reserved to the characterization of limit points of the sequence. Uniqueness of limit points is assumed, since we were not able to prove uniqueness of weak solutions of the partial differential equation (2.10).

4.1. *Tightness*. In this subsection, we present the tightness of $\{\mathbb{Q}^H_{\mu_N}\}$. The proof of this result follows usual ideas and it will be omitted here.

PROPOSITION 4.2. *For fixed $H \in C^{1,2}([0,T]\times[0,1])$, the sequence of measures $\{\mathbb{Q}^H_{\mu_N}; N \ge 1\}$ is tight in the Skorohod topology of $\mathcal{D}_{\mathcal{M}}$.*

4.2. *Radon–Nikodym derivative*. In this section, we deal with the Radon–Nikodym derivative between the SSEP with a slow bond and the WASEP with a slow bond. Its formula will be useful both in the proof of the hydrodynamic limit for the WASEP with a slow bond and in the proof of the large deviations for the SSEP with a slow bond.

By $(\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N})(t)$, we denote the Radon–Nikodym derivative of $\mathbb{P}^H_{\mu_N}$ with respect to $\mathbb{P}_{\mu_N}$ restricted to the σ-algebra generated by $\{\eta_s, 0 \le s \le t\}$. It is a general fact of stochastic processes that $(\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N})(t)$ is a mean-one positive martingale. The explicit formula of the Radon–Nikodym derivative between two Markov process on a countable space state[3] shows that $(\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N})(T)$ is equal to

$$\exp\bigg\{N\bigg[\langle\pi^N_T, H_T\rangle - \langle\pi^N_0, H_0\rangle - \frac{1}{N}\int_0^T e^{-N\langle\pi^N_t, H_t\rangle}(\partial_t + N^2 L_N)e^{N\langle\pi^N_t, H_t\rangle}\,dt\bigg]\bigg\}. \tag{4.1}$$

We are going to write just $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}$ for $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}(T)$, since the time horizon $T > 0$ is fixed. Performing elementary calculations, we can rewrite (4.1) as

$$\begin{aligned}\exp\bigg\{&N\langle\pi^N_T, H_T\rangle - N\langle\pi^N_0, H_0\rangle - N\int_0^T \langle\pi^N_t, \partial_t H_t\rangle\,dt \\ &- N^2\int_0^T \sum_{x\in\mathbb{T}_N} \xi^N_x \eta_t(x)\big(1-\eta_t(x+1)\big)\big(e^{\delta_N H_x} - 1\big)\,dt \\ &- N^2\int_0^T \sum_{x\in\mathbb{T}_N} \xi^N_x \eta_t(x+1)\big(1-\eta_t(x)\big)\big(e^{-\delta_N H_x} - 1\big)\,dt\bigg\}.\end{aligned} \tag{4.2}$$

Since $H \in C^{1,2}([0,T]\times[0,1])$, by Taylor's expansion and the inequality $|e^u - 1 - u - (1/2)u^2| \le (1/6)|u|^3 e^{|u|}$, we observe that all the expressions

- $\frac{1}{N}\sum_{x\neq -1,0} \eta_t(x)N^2(\delta_N H_x - \delta_N H_{x-1}) - \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\partial_u^2 H_t(\frac{x}{N})$,
- $N^2(e^{\pm\delta_N H_x} \mp \delta_N H_x - 1) - \frac{1}{2}(\partial_u H_t)^2(\frac{x}{N})$,
- $N\delta_N H_0 - \partial_u H_t(\frac{0}{N})$ and $N\delta_N H_{-2} - \partial_u H_t(\frac{-1}{N})$

are, in modulus, of order $\frac{1}{N}$. By these facts, we can rewrite (4.2) as

$$\begin{aligned}\exp\bigg\{&N\bigg[\langle\pi^N_T, H_T\rangle - \langle\pi^N_0, H_0\rangle - \int_0^T \langle\pi^N_t, (\partial_t + \Delta)H_t\rangle\,dt \\ &- \int_0^T \bigg\{\eta_t(0)\partial_u H_t\Big(\frac{0}{N}\Big) - \eta_t(-1)\partial_u H_t\Big(\frac{-1}{N}\Big)\bigg\}\,dt + O_{H,T}\Big(\frac{1}{N}\Big)\end{aligned}$$

[3] See Appendix of [9].

$$
\begin{aligned}
&- \int_0^T \frac{1}{N} \sum_{x \neq -1} \big[\eta_t(x)\big(1-\eta_t(x+1)\big) \\
&+ \eta_t(x+1)(1-\eta_t(x)\big]\frac{1}{2}(\partial_u H_t)^2\Big(\frac{x}{N}\Big)\,dt \\
&- \int_0^T \eta_t(-1)\big(1-\eta_t(0)\big)\big(e^{\delta_N H_{-1}}-1\big)\,dt \\
&- \int_0^T \eta_t(0)\big(1-\eta_t(-1)\big)\big(e^{-\delta_N H_{-1}}-1\big)\,dt\bigg]\bigg\}.
\end{aligned}
\tag{4.3}
$$

As we shall see, the expression above is enough in order to prove the hydrodynamical limit of the WASEP with a slow bond. Further estimates on the Radon–Nikodym derivative will be presented at Section 5.

4.3. *Sobolev space*. In this section, we prove that any limit point $\mathbb{Q}_*^H$ of the sequence $\mathbb{Q}_{\mu_N}^H$ is concentrated on trajectories $\rho_t(u)\,du$ belonging the Sobolev space of Definition 2.3. By expression (4.3), there exists a constant $C(H,T)>0$ not depending on N such that

$$
\bigg\|\frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}}\bigg\|_\infty \le \exp\{C(H,T)N\}. \tag{4.4}
$$

It together with Proposition 3.10 implies the next result.

PROPOSITION 4.3. *The measure $\mathbb{Q}_*^H$ is concentrated on paths $\rho_t(u)\,du$ such that $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$.*

4.4. *Characterization of limit points*. Let $\mathbb{Q}_*^H$ be a limit point of the sequence $\{\mathbb{Q}_{\mu_N}^H : N \ge 1\}$ and assume, without loss of generality, that $\{\mathbb{Q}_{\mu_N}^H : N \ge 1\}$ converges to $\mathbb{Q}_*^H$. The existence of $\mathbb{Q}_*^H$ is guaranteed by Proposition 4.2.

PROPOSITION 4.4. *Fix a measurable profile $\rho_0 : \mathbb{T} \to [0,1]$ and consider a sequence $\{\mu_N : N \ge 1\}$ of probability measures on $\{0,1\}^{\mathbb{T}_N}$ associated to ρ_0 in the sense of* (2.2). *Then any limit point of $\mathbb{Q}_{\mu_N}^H$ will be concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)\,du$, with positive density ρ_t bounded by* 1, *such that ρ is a weak solution of* (2.10) *with initial condition ρ_0.*

By Proposition 4.3, we obtain $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$. Then, in order to prove the proposition above, it is enough to assure that $\mathbb{Q}_*^H$ is concentrated in trajectories which satisfy the integral equation (2.12), which is a consequence of Proposition 3.9 together with bound (4.4). This follows the same lines of [4].

5. Large deviations upper bound. The proof of the large deviations upper bound is constructed by an optimization over a class of mean-one positive martingales, which must be functions of the process, or, as in our case, close to functions of the process. In the Section 4.2, we have obtained a good candidate to be the mean-one positive martingale, the Radon–Nikodym derivative of the measure $\mathbb{P}^H_{\mu_N}$ with respect to $\mathbb{P}_{\mu_N}$. Since $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}$ is not a function of the empirical measure, the first step is to show that it is super-exponentially close to a function of the empirical measure.

5.1. *Radon–Nikodym derivative* (*continuation*). To write (4.3) in a simpler form, let us introduce some notation. Given $H \in C^{1,2}([0,T]\times[0,1])$, consider the linear functional $\ell^{\text{int}}_H : \mathcal{D}_{\mathcal{M}} \to \mathbb{R}$

$$\ell^{\text{int}}_H(\pi) = \langle \pi_T, H_T\rangle - \langle \pi_0, H_0\rangle - \int_0^T \langle \pi_t, (\partial_t + \Delta)H_t\rangle\, dt. \tag{5.1}$$

With this notation and recalling (3.4) and (3.5), we can rewrite $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}$ as

$$\begin{aligned}
\exp\Biggl\{ N\Biggl[& \ell^{\text{int}}_H(\pi^N) - \int_0^T \frac{1}{2N}\sum_{x\neq -1}\{\tau_x g_1(\eta_t) + \tau_x g_2(\eta_t)\}(\partial_u H_t)^2\Bigl(\frac{x}{N}\Bigr)\,dt \\
& - \int_0^T \Bigl\{\eta_t(0)\partial_u H_t\Bigl(\frac{0}{N}\Bigr) - \eta_t(-1)\partial_u H_t\Bigl(\frac{-1}{N}\Bigr)\Bigr\}\,dt \\
& - \int_0^T \bigl\{\tau_{-1}g_1(\eta_t)\bigl(e^{\delta_N H_{-1}} - 1\bigr) + \tau_0 g_2(\eta_t)\bigl(e^{-\delta_N H_{-1}} - 1\bigr)\bigr\}\,dt\Biggr] \\
& + N O_{H,T}\Bigl(\frac{1}{N}\Bigr)\Biggr\}.
\end{aligned} \tag{5.2}$$

We begin by defining a set where the Radon–Nikodym derivative $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}$ is close to a function of the empirical measure. Consider

$$W^1_{N,\varepsilon}(t,\eta) := V^{F_1,F_2}_{N,\varepsilon}(t,\eta), \qquad W^2_{N,\varepsilon}(t,\eta) := V^{G_1,G_2}_{N,\varepsilon}(t,\eta),$$
$$W^3_{N,\varepsilon}(t,\eta) := \hat{V}^{\partial_u H,-1}_{N,\varepsilon}(t,\eta), \qquad W^4_{N,\varepsilon}(t,\eta) := \hat{V}^{\partial_u H,0}_{N,\varepsilon}(t,\eta),$$

where V and $\hat{V}$ have been defined in Proposition 3.4 and Corollary 3.6 considering $F_1(t,u) = \frac{1}{2}(\partial_u H_t)^2(u)$, $F_2(t,\frac{-1}{N}) = e^{\delta_N H_{-1}} - 1$, $G_1(t,u) = \frac{1}{2}(\partial_u H_t)^2(u)$ and $G_2(t,\frac{-1}{N}) = e^{-\delta_N H_{-1}} - 1$. Define the set

$$B^H_{\delta,\varepsilon} = \biggl\{\eta \in \mathcal{D}_{\Omega_N};\ \biggl|\int_0^T W^i_{N,\varepsilon}(t,\eta_t)\,dt\biggr| \le \delta, i = 1,2,3,4\biggr\}. \tag{5.3}$$

From Proposition 3.4 and Corollary 3.6, this set $B^H_{\delta,\varepsilon}$ has probability superexponentially close to one, that is, for each $\delta > 0$,

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log \mathbb{P}_{\mu_N}\bigl[(B^H_{\delta,\varepsilon})^{\complement}\bigr] = -\infty. \tag{5.4}$$

In view of identity (3.3) and expression (5.2), restricted to the set $B^H_{\delta,\varepsilon}$ the Radon–Nikodym derivative $\mathbf{d}\mathbb{P}^H_{\mu_N}/\mathbf{d}\mathbb{P}_{\mu_N}$ is equal to

$$
\begin{aligned}
\exp\bigg\{ N\bigg[& \ell^{\text{int}}_H(\mathcal{A}) + O_{H,T}\Big(\frac{1}{N}\Big) + O(\delta) \\
& - \int_0^T \frac{1}{2N}\sum_{x\neq -1}\Big\{\tilde{g}_1\Big(\mathcal{A}\Big(\frac{x}{N}\Big),\mathcal{A}\Big(\frac{x+1}{N}\Big)\Big) \\
& + \tilde{g}_2\Big(\mathcal{A}\Big(\frac{x}{N}\Big),\mathcal{A}\Big(\frac{x+1}{N}\Big)\Big)\Big\}(\partial_u H_t)^2\Big(\frac{x}{N}\Big)\,dt \\
& - \int_0^T \Big[\mathcal{A}\Big(\frac{0}{N}\Big)\partial_u H_t\Big(\frac{0}{N}\Big) - \mathcal{A}\Big(\frac{-1}{N}\Big)\partial_u H_t\Big(\frac{-1}{N}\Big)\Big]\,dt \\
& - \int_0^T \tilde{g}_1\Big(\mathcal{A}\Big(\frac{-1}{N}\Big),\mathcal{A}\Big(\frac{0}{N}\Big)\Big)(e^{\delta_N H_{-1}}-1)\,dt \\
& - \int_0^T \tilde{g}_2\Big(\mathcal{A}\Big(\frac{-1}{N}\Big),\mathcal{A}\Big(\frac{0}{N}\Big)\Big)(e^{-\delta_N H_{-1}}-1)\,dt\bigg]\bigg\},
\end{aligned}
\tag{5.5}
$$

where $\mathcal{A}=\pi^N_t * \iota_\varepsilon$. At this point, we have a function of the empirical measure modulo some small errors. Unfortunately, this is not enough to handle with limits on boundary terms. The reason is simple, the convolution $\pi^N * \iota_\varepsilon$ is a function (not a measure anymore) but not a *smooth* function, therefore, not necessarily possessing well-behaved side limits. Hence, the next step is to replace $\pi^N * \iota_\varepsilon$ by $(\pi^N * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon$, where $\iota^{\mathrm{s}}_\gamma$ is a smooth approximation of identity to be defined next. Notice that $\iota^{\mathrm{s}}_\gamma$ shall not be misunderstood with ι_ε defined in (3.2).

Fix $f:\mathbb{T}\to\mathbb{R}_+$ a continuous function with support contained in $[-\frac{1}{4},\frac{1}{4}]$, $0\leq f\leq 4$, $f(0)>0$, $\int f=1$ and symmetric around zero, in other words, satisfying $f(u)=f(1-u)$ for all $u\in\mathbb{T}$. Define the continuous approximation of identity $\iota^{\mathrm{s}}_\gamma$ by $\iota^{\mathrm{s}}_\gamma(u)=\frac{1}{\gamma}f(\frac{u}{\gamma})$.

At this point, we need some approximation estimates to be presented in three next lemmas. Its proofs can be found in the extended version [8]. Recall that ℓ^{int}_H is the linear functional defined in (5.1).

LEMMA 5.1. $|(\pi^N_t * \iota_\varepsilon)(v) - ((\pi^N_t * \iota^s_\gamma) * \iota_\varepsilon)(v)| \leq \frac{\gamma}{\varepsilon}$, *uniformly in* $v\in\mathbb{T}$, $N\in\mathbb{N}$, *and* $t\in[0,T]$.

LEMMA 5.2. $\ell^{\text{int}}_H(\pi^N) = \ell^{\text{int}}_H((\pi^N * \iota^s_\gamma) * \iota_\varepsilon) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon})$, *uniformly in* $N\in\mathbb{N}$.

LEMMA 5.3. *The function* $|\tilde{g}_i((\pi^N_t * \iota_\varepsilon)(\frac{x}{N}), (\pi^N_t * \iota_\varepsilon)(\frac{x+1}{N})) - \tilde{g}_i(((\pi^N_t * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)(\frac{x}{N}), ((\pi^N_t * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)(\frac{x+1}{N}))|$ *is* $O(\frac{\gamma}{\varepsilon})$ *for* $i=1,2$.

Lemmas 5.1, 5.2 and 5.3 allow to replace π_t^N by $(\pi_t^N * \iota_\gamma^{\mathrm{s}})$ in the expression of Radon–Nikodym derivative (5.5) modulus small errors. Hence, restricted to the set $B_{\delta,\varepsilon}^H$, the Radon–Nikodym derivative $\mathbf{d}\mathbb{P}_{\mu_N}^H/\mathbf{d}\mathbb{P}_{\mu_N}$ becomes

$$
\begin{aligned}
\exp\Bigg\{N\Bigg[&\ell_H^{\mathrm{int}}(\mathcal{B}) + O_{H,T}\Big(\frac{1}{N}\Big) + O(\delta) + O_H(\varepsilon) + O_H\Big(\frac{\gamma}{\varepsilon}\Big)\\
&- \int_0^T \frac{1}{2N}\sum_{x\neq -1}\Big\{\tilde{g}_1\Big(\mathcal{B}\Big(\frac{x}{N}\Big), \mathcal{B}\Big(\frac{x+1}{N}\Big)\Big)\\
&+ \tilde{g}_2\Big(\mathcal{B}\Big(\frac{x}{N}\Big), \mathcal{B}\Big(\frac{x+1}{N}\Big)\Big)\Big\}(\partial_u H_t)^2\Big(\frac{x}{N}\Big)\,dt\\
&- \int_0^T \Big[\mathcal{B}\Big(\frac{0}{N}\Big)\partial_u H_t\Big(\frac{0}{N}\Big) - \mathcal{B}\Big(\frac{-1}{N}\Big)\partial_u H_t\Big(\frac{-1}{N}\Big)\Big]\,dt\\
&- \int_0^T \tilde{g}_1\Big(\mathcal{B}\Big(\frac{-1}{N}\Big), \mathcal{B}\Big(\frac{0}{N}\Big)\Big)\big(e^{\delta_N H_{-1}} - 1\big)\,dt\\
&- \int_0^T \tilde{g}_2\Big(\mathcal{B}\Big(\frac{-1}{N}\Big), \mathcal{B}\Big(\frac{0}{N}\Big)\Big)\big(e^{-\delta_N H_{-1}} - 1\big)\,dt\Bigg]\Bigg\},
\end{aligned}
\tag{5.6}
$$

where $\mathcal{B} = (\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon$. The next three lemmas allow to replace the sum involving $\tilde{g}_i$ by an integral in χ and to make a little adjustment at the boundaries. Its proofs will be omitted as well.

LEMMA 5.4. *The difference*

$$
\begin{aligned}
\Bigg|\frac{1}{N}\sum_{x\neq -1}\tilde{g}_i\Big(((\pi_t^N * \iota_\gamma^{s}) * \iota_\varepsilon)\Big(\frac{x}{N}\Big), ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)\Big(\frac{x+1}{N}\Big)\Big)(\partial_u H_t)^2\Big(\frac{x}{N}\Big)&\\
- \int_{\mathbb{T}} \chi(((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)(v))(\partial_u H_t)^2(v)\,dv\Bigg|&,
\end{aligned}
$$

can be denoted by some function $R_N^1(H, t, \varepsilon, \gamma)$, which goes to zero, when $N \to \infty$, uniformly in $t \in [0, T]$, with $i = 1, 2$.

LEMMA 5.5. *Denote by $R_N^2(H, t, \varepsilon, \gamma)$ the following expression*:

$$
\begin{aligned}
\Bigg|((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)\Big(\frac{0}{N}\Big)\partial_u H_t\Big(\frac{0}{N}\Big) - ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)\Big(\frac{-1}{N}\Big)\partial_u H_t\Big(\frac{-1}{N}\Big)&\\
- ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)(0^+)\partial_u H_t(0^+) - ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)(0^-)\partial_u H_t(0^-)\Bigg|&.
\end{aligned}
$$

Then $R_N^2(H, t, \varepsilon, \gamma)$ goes to zero, when N increases to ∞, uniformly in $t \in [0, T]$.

LEMMA 5.6. *The expression below*

$$\Big|\tilde{g}_1(((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)(0^-), ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)(0^+))(e^{\delta H_t(0)} - 1) - \tilde{g}_1\left(((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)\left(\frac{-1}{N}\right), ((\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon)\left(\frac{0}{N}\right)\right)(e^{\delta_N H_{-1}} - 1)\Big|$$

is a function $R_N^3(H, t, \varepsilon, \gamma)$, which goes to zero, when N increases to ∞, uniformly in $t \in [0, T]$. Analogous statement for $\tilde{g}_2$.

Denote $R_N(H, T, \varepsilon, \gamma)$ the errors from the Lemmas 5.4, 5.5 and 5.6. Notice that

$$\lim_{N\to\infty} R_N(H, T, \varepsilon, \gamma) = 0. \tag{5.7}$$

By means of these lemmas, we can rewrite the expression (5.6) of the Radon–Nikodyn derivative $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ on the set $B_{\delta,\varepsilon}^H$ as

$$\begin{aligned}\exp\Big\{N\Big[&\ell_H^{\mathrm{int}}(\mathcal{B}) - \int_0^T \int_{\mathbb{T}} \chi(\mathcal{B}(v))(\partial_u H_t)^2(v)\, dv\, dt \\ &- \int_0^T [\mathcal{B}(0^+)\partial_u H_t(0^+) - \mathcal{B}(0^-)\partial_u H_t(0^-)]\, dt \\ &- \int_0^T \tilde{g}_1(\mathcal{B}(0^-), \mathcal{B}(0^+))(e^{\delta H_t(0)} - 1)\, dt \\ &- \int_0^T \tilde{g}_2(\mathcal{B}(0^-), \mathcal{B}(0^+))(e^{-\delta H_t(0)} - 1)\, dt \\ &+ R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right)\Big]\Big\},\end{aligned} \tag{5.8}$$

where $\mathcal{B} = (\pi_t^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon$ as before.

Now we observe that the functional ℓ_H defined in (2.15) and the functional ℓ_H^{int} given in Definition (5.1) are related by

$$\begin{aligned}\ell_H(\pi) = \ell_H^{\mathrm{int}}(\pi) &- \int_0^T \{\rho_t(0^+)\partial_u H_t(0^+) - \rho_t(0^-)\partial_u H_t(0^-)\}\, dt \\ &+ \int_0^T (\rho_t(0^+) - \rho_t(0^-))(H_t(0^+) - H_t(0^-))\, dt.\end{aligned}$$

Moreover, because of its smoothness, $(\pi^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon$ has finite energy; see Definition 2.9. Recalling Definition 2.10 of the functional J_H, and expression (5.8), we conclude that $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ restricted to $B_{\delta,\varepsilon}^H$ is

$$\begin{aligned}\exp\Big\{N\Big[&J_H((\pi^N * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon) + R_N(H, T, \varepsilon, \gamma) \\ &+ O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right)\Big]\Big\}.\end{aligned} \tag{5.9}$$

Let us proceed to the next step. It is not difficult to see that the set $\{\pi \in \mathcal{D}_{\mathcal{M}}; \mathcal{E}(\pi) < \infty\}$ is not closed in the concerning topology (the Skorohod topology on $\mathcal{D}_{\mathcal{M}}$). This is an obstacle to apply the minimax lemma; see [9], Lemma 3.3, page 364, which is an important device in the proof of the large deviations upper bound. To invoke the minimax lemma, the functional J_H should be lower semicontinuous,[4] what is not true precisely because the set $\{\pi \in \mathcal{D}_{\mathcal{M}}; \mathcal{E}(\pi) < \infty\}$ is not closed.

To overcome this obstacle, we begin by introducing the next sets.

DEFINITION 5.7. Let $A_{k,l}$, $A^{\varepsilon}_{k,l}$, and $A^{\varepsilon,\gamma}_{k,l}$ be the subsets of trajectories given by

$$\begin{aligned} A_{k,l} &= \Big\{\pi \in \mathcal{D}_{\mathcal{M}}; \max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi) \leq l\Big\},\\ A^{\varepsilon}_{k,l} &= \{\pi \in \mathcal{D}_{\mathcal{M}}; \pi * \iota_\varepsilon \in A_{k,l}\},\\ A^{\varepsilon,\gamma}_{k,l} &= \big\{\pi \in \mathcal{D}_{\mathcal{M}}; \big(\pi * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon \in A_{k,l}\big\}. \end{aligned}$$

PROPOSITION 5.8. *For fixed $\varepsilon, \gamma, k, l$, the set $A^{\varepsilon,\gamma}_{k,l}$ is closed.*

PROOF. It is sufficient to show that the function $\psi : \mathcal{D}_{\mathcal{M}} \to \overline{\mathbb{R}}$ given by $\psi(\pi) = \mathcal{E}_{H_j}((\pi^N * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)$ is continuous. Let $\{\pi^n_t; t \in [0,T]\}_n$ converging to $\{\pi_t; t \in [0,T]\}$ on $\mathcal{D}_{\mathcal{M}}$. Therefore, $\pi^n_t \stackrel{\omega^*}{\to} \pi_t$, almost surely in time. For such t, $\pi_t * \iota^{\mathrm{s}}_\gamma = \lim_{n\to\infty} \pi^n_t * \iota^{\mathrm{s}}_\gamma$, since $\iota^{\mathrm{s}}_\gamma$ is a continuous function. By the dominated convergence theorem,

$$\begin{aligned} \big(\big(\pi_t * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon\big)(v) &= \int_{\mathbb{T}} \lim_{n\to\infty} \big(\pi^n_t * \iota^{\mathrm{s}}_\gamma\big)(u)\iota_\varepsilon(u,v)\,du\\ &= \lim_{n\to\infty} \big(\big(\pi^n_t * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon\big)(v). \end{aligned} \tag{5.10}$$

Again by the dominated convergence theorem,

$$\begin{aligned} \langle\!\langle \partial_u H_j, \big(\pi_t * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon \rangle\!\rangle &= \int_0^T \int_{\mathbb{T}} \partial_u H_j(t,v)\big(\big(\pi_t * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon\big)(v)\,dv\,dt\\ &= \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}} \partial_u H_j(t,v)\big(\big(\pi^n_t * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon\big)(v)\,dv\,dt\\ &= \lim_{n\to\infty} \langle\!\langle \partial_u H_j, \big(\pi^n * \iota^{\mathrm{s}}_\gamma\big) * \iota_\varepsilon \rangle\!\rangle. \end{aligned}$$

□

[4]About signs and conventions: in [9], Lemma 3.3, page 364, the statement is about an upper continuous functional, but the functional J_β appearing there corresponds to minus our functional J_H here.

PROPOSITION 5.9. *For fixed k and l,*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{\gamma\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\mu_N}\big[\pi^N\in\big(A_{k,l}^{\varepsilon,\gamma}\big)^{\complement}\big]\le -l+K_0T.$$

PROOF. For all $r>0$,

$$\begin{aligned}
&\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon\big)\ge l\Big]\\
&\quad\le \mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big(\pi^N*\iota_\varepsilon\big)\ge l-r\Big]\\
&\qquad+\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon-\pi^N*\iota_\varepsilon\big)\ge r\Big].
\end{aligned}$$

By Lemma 5.1, we have that

$$\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon-\pi^N*\iota_\varepsilon\big)\le\max_{1\le j\le k}\langle\!\langle\partial_u H_j,(\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon-\pi^N*\iota_\varepsilon\rangle\!\rangle\le\frac{C\gamma}{\varepsilon},$$

where $C=C(\{H\}_{1\le j\le k})$. Therefore,

$$\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}}-\pi^N)*\iota_\varepsilon\big)\ge r\Big]\le\mathbb{P}_{\mu_N}\Big[\frac{C\gamma}{\varepsilon}\ge r\Big],$$

which is zero for γ small enough. Hence,

$$\begin{aligned}
&\varlimsup_{\gamma\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon\big)\ge l\Big]\\
&\quad\le\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big(\pi^N*\iota_\varepsilon\big)\ge l-r\Big].
\end{aligned}$$

By Corollary 3.8, we get

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{\gamma\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}\big((\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon\big)\ge l\Big]\le -l+K_0T+r.$$

Since r is arbitrary, the proof is complete. □

In (5.9) we use the term $(\pi^N*\iota_\gamma^{\mathrm{s}})*\iota_\varepsilon$ and we would like to take $\gamma\downarrow 0$ and $\varepsilon\downarrow 0$. To avoid technical problems that would come into scene from the fact π_t^N does not have density with respect to the Lebesgue measure, we define below another family of sets.

Fix a sequence $\{F_i\}_{i\ge 1}$ of smooth nonnegative functions dense in the subset of nonnegative functions $C(\mathbb{T})$ with respect to the uniform topology. For $i\ge 1$ and $j\ge 1$, we define the set

$$D_i^j=\Big\{\pi\in\mathcal{D}_{\mathcal{M}};0\le\langle\pi_t,F_i\rangle\le\int_{\mathbb{T}}F_i(u)\,du+\frac{1}{j}\|F_i'\|_\infty,0\le t\le T\Big\},\tag{5.11}$$

and for $m\ge 1$ and $j\ge 1$, let $E_m^j=\bigcap_{i=1}^m D_i^j$.

PROPOSITION 5.10. *It holds that*:

(i) *Given* $i \geq 1$ *and* $j \geq 1$, *the set* D_i^j *is a closed subset of* $\mathcal{D}_{\mathcal{M}}$.

(ii) $\mathcal{D}_{\mathcal{M}_0} = \bigcap_{j\geq 1}\bigcap_{m\geq 1} E_m^j$.

(iii) *Given* $m \geq 1$ *and* $j \geq 1$, $\overline{\lim}_{N\to\infty} \frac{1}{N}\log \mathbb{P}_{\mu_N}[\pi^N \in (E_m^j)^{\complement}] = -\infty$.

The proof of proposition above, which is similar to that one of [3], Section 6.3, can be found in [8].

Keeping in mind that $\mathcal{E}((\pi * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon) < \infty$, for all $\pi \in \mathcal{D}_{\mathcal{M}}$, define

$$J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi) = \begin{cases} \hat{J}_H\big((\pi * \iota_\gamma^{\mathrm{s}}) * \iota_\varepsilon\big) & \text{if } \pi \in A_{k,l}^{\zeta,\gamma} \cap E_m^j, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.12}$$

Finally, $\mathbf{d}\mathbb{P}_{\mu_N}^H/\mathbf{d}\mathbb{P}_{\mu_N}$ restricted to the set $\{\pi^N \in A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H$ is

$$\begin{aligned} \exp\bigg\{ N\bigg[& J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi^N) + R_N(H,T,\varepsilon,\gamma) \\ & + O(\delta) + O_H(\varepsilon) + O_H\Big(\frac{\gamma}{\varepsilon}\Big)\bigg]\bigg\}. \end{aligned} \tag{5.13}$$

This is the appropriate form for the Radon–Nikodym derivative to be used in the next section.

5.2. *Upper bound for compact sets*. We start by studying the upper bound for open sets. Let $\mathcal{O} \subseteq \mathcal{D}_{\mathcal{M}}$ be an open set and fix a function $H \in C^{1,2}([0,T]\times[0,1])$. Then

$$\begin{aligned} \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{Q}_{\mu_N}[\mathcal{O}] &= \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_{\mu_N}[\pi^N \in \mathcal{O}] \\ &\leq \max\bigg\{ \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_{\mu_N}[\{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H], \\ &\qquad R_k^l(\zeta,\gamma), R_m^j, R_H^\delta(\varepsilon)\bigg\}, \end{aligned}$$

where we have denoted

$$\begin{aligned} R_k^l(\zeta,\gamma) &= \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_{\mu_N}\big[\{\pi^N \in \big(A_{k,l}^{\zeta,\gamma}\big)^{\complement}\}\big], \\ R_m^j &= \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_{\mu_N}\big[\{\pi^N \in \big(E_m^j\big)^{\complement}\}\big], \\ R_H^\delta(\varepsilon) &= \overline{\lim_{N\to\infty}} \frac{1}{N}\log \mathbb{P}_{\mu_N}\big[\big(B_{\delta,\varepsilon}^H\big)^{\complement}\big]. \end{aligned}$$

By Propositions 5.9 and 5.10 and the limit (5.4), the expressions above satisfy

$$\overline{\lim_{\zeta\downarrow 0}}\,\overline{\lim_{\gamma\downarrow 0}} R_k^l(\zeta,\gamma) \leq -l + K_0 T, \qquad R_m^j = -\infty, \quad \text{and} \quad \overline{\lim_{\varepsilon\downarrow 0}} R_H^\delta(\varepsilon) = -\infty.$$

Transforming the measure by the Radon–Nikodym derivative and recalling its expression (5.13),

$$
\begin{aligned}
&\mathbb{P}_{\mu_N}\big[\{\pi^N \in \mathcal{O}\cap A_{k,l}^{\zeta,\gamma}\cap E_m^j\}\cap B_{\delta,\varepsilon}^H\big]\\
&\quad= \mathbb{E}_{\mu_N}^H\bigg[\bigg(\frac{\mathrm{d}\mathbb{P}_{\mu_N}^H}{\mathrm{d}\mathbb{P}_{\mu_N}}\bigg)^{-1}\mathbf{1}_{\{\pi^N\in\mathcal{O}\cap A_{k,l}^{\zeta,\gamma}\cap E_m^j\}\cap B_{\delta,\varepsilon}^H}\bigg]\\
&\quad= \mathbb{E}_{\mu_N}^H\bigg[\exp\bigg\{N\bigg[-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi^N)+R_N(H,T,\varepsilon,\gamma)\\
&\qquad\qquad + O(\delta)+O_H(\varepsilon)+O_H\Big(\frac{\gamma}{\varepsilon}\Big)\bigg]\bigg\}\mathbf{1}_{\mathbf{D}}\bigg],
\end{aligned}
$$

with $\mathbf{D} := \{\pi^N \in \mathcal{O}\cap A_{k,l}^{\zeta,\gamma}\cap E_m^j\}\cap B_{\delta,\varepsilon}^H$. Therefore,

$$
\begin{aligned}
&\frac{1}{N}\log\mathbb{P}_{\mu_N}\big[\{\pi^N \in \mathcal{O}\cap A_{k,l}^{\zeta,\gamma}\cap E_m^j\}\cap B_{\delta,\varepsilon}^H\big]\\
&\qquad\le \sup_{\pi\in\mathcal{O}}\{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\}+R_N(H,T,\varepsilon,\gamma)+O(\delta)+O_H(\varepsilon)+O_H\Big(\frac{\gamma}{\varepsilon}\Big).
\end{aligned}
$$

By (5.7), for all $\gamma,\varepsilon,\zeta,\delta>0$, for all $k,l,m,j\in\mathbb{N}$ and $H\in C^{1,2}([0,T]\times[0,1])$, we have

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[\mathcal{O}]\\
&\quad\le \max\bigg\{\sup_{\pi\in\mathcal{O}}\{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\}+O(\delta)\\
&\qquad\qquad + O_H(\varepsilon)+O_H\Big(\frac{\gamma}{\varepsilon}\Big), R_k^l(\zeta,\gamma), R_m^j, R_H^\delta(\varepsilon)\bigg\}\\
&\quad= \max\bigg\{\sup_{\pi\in\mathcal{O}}\{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\}+O(\delta)+O_H(\varepsilon)\\
&\qquad\qquad + O_H\Big(\frac{\gamma}{\varepsilon}\Big), R_k^l(\zeta,\gamma), R_H^\delta(\varepsilon)\bigg\}.
\end{aligned}
$$

Since we do not have any restrictions on the parameters, we can optimize over $\gamma,\varepsilon,\zeta,\delta,k,l,m,j,H$, which yields

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[\mathcal{O}]\\
&\qquad\le \inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}}\max\bigg\{\sup_{\pi\in\mathcal{O}}\{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\}\\
&\qquad\qquad + O(\delta)+O_H(\varepsilon)+O_H\Big(\frac{\gamma}{\varepsilon}\Big), R_k^l(\zeta,\gamma), R_H^\delta(\varepsilon)\bigg\}
\end{aligned}
\tag{5.14}
$$

$$= \inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}} \sup_{\pi\in\mathcal{O}} \max\Big\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\Big(\frac{\gamma}{\varepsilon}\Big), R^l_k(\zeta,\gamma), R^\delta_H(\varepsilon)\Big\}.$$

PROPOSITION 5.11. *For fixed $\gamma,\varepsilon,\zeta,\delta,k,l,m,j,H$, the functional*

$$\max\Big\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\Big(\frac{\gamma}{\varepsilon}\Big), R^l_k(\zeta,\gamma), R^\delta_H(\varepsilon)\Big\}$$

is upper semicontinuous in $\mathcal{D}_{\mathcal{M}}$.

PROOF. In the maximum above, the only term that depends on π is $J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)$. By the Propositions 5.8 and 5.10, it is enough to prove the continuity of $\hat{J}((\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)$ in $\mathcal{D}_{\mathcal{M}}$.

Let $\pi^n \to \pi$ in the topology of $\mathcal{D}_{\mathcal{M}}$. In particular, π^n_t converges weakly* to π_t in $\mathcal{M}$, for almost all $t \in [0,T]$. According to (5.10) and iterated applications of the dominated convergence theorem, we can assure the continuity of $\hat{J}((\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)$. $\square$

Provided by the proposition above, we may apply the minimax lemma [9], Lemma A2.3.3, interchanging supremum with infimum in (5.14), and passing to compacts sets. Then, for all $\mathcal{K} \subset \mathcal{D}_{\mathcal{M}}$ compact,

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \\
&\quad\le \sup_{\pi\in\mathcal{K}} \inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}} \max\Big\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) \\
&\qquad + O_H\Big(\frac{\gamma}{\varepsilon}\Big), R^l_k(\zeta,\gamma), R^\delta_H(\varepsilon)\Big\}.
\end{aligned} \tag{5.15}$$

The next result connects $J_H(\pi)$ and $J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)$.

PROPOSITION 5.12. *For all $\pi \in \mathcal{D}_{\mathcal{M}}$,*

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{l\to\infty} \varlimsup_{k\to\infty} \varlimsup_{\zeta\downarrow 0} \varlimsup_{\gamma\downarrow 0} \varlimsup_{j\to\infty} \varlimsup_{m\to\infty} J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) \ge J_H(\pi).$$

PROOF. Recall (5.12) and fix $\pi \in \mathcal{D}_{\mathcal{M}}$. We claim that

$$\varlimsup_{j\to\infty} \varlimsup_{m\to\infty} J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) = \begin{cases} \hat{J}_H\big((\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big) & \text{if } \pi \in A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}_{\mathcal{M}_0}, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.16}$$

The equality above derives from the fact that if $\pi \notin \mathcal{D}_{\mathcal{M}_0}$, there exist m and j such that $\pi \notin E_m^j$. To check this, apply the definition of an absolute continuity with respect to the Lebesgue measure. This proves (5.16).

Let us step to the limit in γ. We claim that

$$
\begin{aligned}
\varlimsup_{\gamma\downarrow 0}
\begin{cases}
\hat{J}_H\big((\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big) & \text{if } \pi \in A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}_{\mathcal{M}_0}\\
+\infty & \text{otherwise}
\end{cases}\\
\geq
\begin{cases}
\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \pi \in A^{\zeta}_{k,l+1} \cap \mathcal{D}_{\mathcal{M}_0},\\
+\infty & \text{otherwise.}
\end{cases}
\end{aligned}
\tag{5.17}
$$

If $\pi \notin A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}_{\mathcal{M}_0}$ for all γ, the inequality (5.17) is obvious. From Definition 5.7, if $\pi \in A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}_{\mathcal{M}_0}$, it is immediate that

$$
\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) \leq l + \max_{1\leq j\leq k} \langle\!\langle \partial_u H_j, \pi * \iota_\zeta - \big(\pi * \iota^{\mathrm{s}}_\gamma\big) * \iota_\zeta \rangle\!\rangle.
$$

For fixed ζ and k, we can find γ small enough in such a way

$$
\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) \leq l+1,
$$

implying $\pi \in A^{\zeta}_{k,l+1} \cap \mathcal{D}_{\mathcal{M}_0}$. Besides, for fixed $\varepsilon > 0$, the double convolution $(\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon$ converges uniformly to $\pi * \iota_\varepsilon$, leading to

$$
\lim_{\gamma\downarrow 0} \hat{J}_H\big((\pi * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big) = \hat{J}_H(\pi * \iota_\varepsilon)
$$

and hence proves (5.17). The ensuing step is to take the limit in $\zeta \downarrow 0$. We claim that

$$
\begin{aligned}
\varlimsup_{\zeta\downarrow 0}
\begin{cases}
\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \pi \in A^{\zeta}_{k,l+1} \cap \mathcal{D}_{\mathcal{M}_0}\\
+\infty & \text{otherwise}
\end{cases}\\
\geq
\begin{cases}
\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \pi \in A_{k,l+2} \cap \mathcal{D}_{\mathcal{M}_0},\\
+\infty & \text{otherwise.}
\end{cases}
\end{aligned}
\tag{5.18}
$$

In fact, if $\pi \in A^{\zeta}_{k,l+1} \cap \mathcal{D}_{\mathcal{M}_0}$, then

$$
\begin{aligned}
\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi) &= \max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) + \max_{1\leq j\leq k} \langle\!\langle \partial_u H_j, \pi - \pi * \iota_\zeta \rangle\!\rangle\\
&\leq l + 1 + \max_{1\leq j\leq k} \int_0^T \int_{\mathbb{T}} \partial_u H_j(t,u)\big(\rho_t(u) - (\pi_t * \iota_\zeta)(u)\big)\, du\, dt.
\end{aligned}
$$

By the Lebesgue differentiation theorem, it is possible to choose small ζ such that the integral term in the right-hand side of above is smaller than 1. This proves

(5.18). Taking the limit in $k \to \infty$ in the right-hand side of (5.18), we obtain

$$\begin{aligned}\overline{\lim_{k\to\infty}} &\begin{cases}\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \pi \in A_{k,l+2} \cap \mathcal{D}_{\mathcal{M}_0}\\ +\infty & \text{otherwise}\end{cases}\\ &= \begin{cases}\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \mathcal{E}(\pi) \le l+2,\\ +\infty & \text{otherwise,}\end{cases}\end{aligned} \tag{5.19}$$

because $\{\pi; \mathcal{E}(\pi) \le l+2\} \subset \mathcal{D}_{\mathcal{M}_0}$. Next, taking the limit in $l \to \infty$ in the right hand side of (5.19), we get

$$\overline{\lim_{l\to\infty}} \begin{cases}\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \mathcal{E}(\pi) \le l+2\\ +\infty & \text{otherwise}\end{cases} \ge \begin{cases}\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \mathcal{E}(\pi) < \infty,\\ +\infty & \text{otherwise.}\end{cases}$$

Finally, taking the limit when $\varepsilon \downarrow 0$ in the right-hand side of above, it yields

$$\overline{\lim_{\varepsilon\downarrow 0}} \begin{cases}\hat{J}_H(\pi * \iota_\varepsilon) & \text{if } \mathcal{E}(\pi) < \infty\\ +\infty & \text{otherwise}\end{cases} = J_H(\pi),$$

where we have used that, for $\pi \in \{\pi; \mathcal{E}(\pi) < \infty\}$ it holds that $\pi_t(du) = \rho_t(u)\,du$, where ρ has well-defined left and right side limits around zero. $\square$

PROPOSITION 5.13 (Upper bound for compact sets). *For every $\mathcal{K}$ compact subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \le -\inf_{\pi\in\mathcal{K}} I(\pi).$$

PROOF. Proposition 5.12 can be restated in the form

$$\underline{\lim}_{\varepsilon\downarrow 0}\, \underline{\lim}_{l\to\infty}\, \underline{\lim}_{k\to\infty}\, \underline{\lim}_{\zeta\downarrow 0}\, \underline{\lim}_{\gamma\downarrow 0}\, \underline{\lim}_{j\to\infty}\, \underline{\lim}_{m\to\infty} -J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) \le -J_H(\pi),$$

for all $\pi \in \mathcal{D}_{\mathcal{M}}$. Plugging this into (5.15) leads to

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \le \sup_{\pi\in\mathcal{K}} \inf_H \{-J_H(\pi)\} = -\inf_{\pi\in\mathcal{K}} \sup_H J_H(\pi) = -\inf_{\pi\in\mathcal{K}} I(\pi).$$

$\square$

5.3. *Upper bound for closed sets.*

PROPOSITION 5.14 (Upper bound for closed sets). *For every $\mathcal{C}$ closed subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{C}] \le -\inf_{\pi\in\mathcal{C}} I(\pi).$$

By exponential tightness, we mean that there exists compact sets $K_n \subset \mathcal{D}_{\mathcal{M}}$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[K_n^\complement] \leq -n, \qquad \forall n \in \mathbb{N}.$$

It is well known that the upper bound for closed sets is an immediate consequence of upper bound for compact sets plus exponential tightness. The rest of this section is concerned which exponential tightness, which we affirm it is a consequence of next lemma.

LEMMA 5.15. *For* $\varepsilon > 0$, $\delta > 0$ *and* $H \in C^2(\mathbb{T})$, *denote*

$$\mathcal{C}_{H,\delta,\varepsilon} := \Big\{\pi \in \mathcal{D}_{\mathcal{M}};\ \sup_{s\leq t\leq s+\delta} \big|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\big| \leq \varepsilon, \forall s \in [0,T]\Big\}.$$

Then, for every $\varepsilon > 0$ *and every function* $H \in C^2(\mathbb{T})$, *the following limit holds*:

$$\lim_{\delta\downarrow 0} \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\pi \notin \mathcal{C}_{H,\delta,\varepsilon}] = -\infty.$$

Indeed, suppose the statement above. Let $\{H_\ell\}_{\ell\in\mathbb{N}} \subset C^2(\mathbb{T})$ be a dense set of functions in $C(\mathbb{T})$ for the uniform topology. For each $\delta > 0$ and $\ell, m \in \mathbb{N}$, denote by $C_{\ell,\delta,\frac{1}{m}}$ the set $\mathcal{C}_{H_\ell,\delta,\varepsilon}$ with $\varepsilon = \frac{1}{m}$. Assuming Lemma 5.15, in particular we have that

$$\lim_{\delta\downarrow 0} \overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta,\frac{1}{m}}] = -\infty, \qquad \forall \ell, m \geq 1. \tag{5.20}$$

Fix positive integers ℓ, m. In view of (5.20), for any $n \in \mathbb{N}$ we can find $\delta_0 = \delta_0(\ell, m, n) > 0$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta,\frac{1}{m}}] \leq -nm\ell, \qquad \forall \delta \in (0, \delta_0].$$

Hence, for each $\delta \in (0, \delta_0]$ there exists $N_\delta = N_\delta(\delta, \ell, m, n) \in \mathbb{N}$ such that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta,\frac{1}{m}}] \leq e^{-Nnm\ell}, \qquad \forall N \geq N_\delta.$$

At this point, some efforts are necessary in order to remove the restriction above on N (by suitably re-defining δ). This is the content of the claim:

Claim: For all positive integers ℓ, m, n, there exists $\tilde{\delta} = \tilde{\delta}(\ell, m, n) > 0$ such that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\tilde{\delta},\frac{1}{m}}] \leq e^{-Nnm\ell}, \qquad \forall N \in \mathbb{N}.$$

To prove this claim, we start by observing that, if $0 < \delta_1 < \delta_2$, then $C_{\ell,\delta_2,\frac{1}{m}} \subseteq C_{\ell,\delta_1,\frac{1}{m}}$. Hence,

$$[\pi \notin C_{\ell,\delta_1,\frac{1}{m}}] \subseteq [\pi \notin C_{\ell,\delta_2,\frac{1}{m}}], \qquad \text{for } 0 < \delta_1 < \delta_2. \tag{5.21}$$

Now, denoting $N_0 = N_{\delta_0}(\ell, m, n)$ (which depends only on ℓ, m, n, because δ_0 is a function of ℓ, m, n), we have that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta,\frac{1}{m}}] \leq \mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta_0,\frac{1}{m}}] \leq e^{-Nnm\ell}, \tag{5.22}$$

$\forall \delta \in (0, \delta_0]$ and $\forall N \geq N_0$.

Observe that, for fixed $\ell, m \in \mathbb{N}$, we have that $C_{\ell,\delta,\frac{1}{m}} \nearrow \mathcal{D}_{\mathcal{M}}$ as $\delta \searrow 0$, which is true because the set $\mathcal{D}_{\mathcal{M}}$ is composed of càdlàg trajectories. Since the sets $[\pi \notin C_{\ell,\delta,\frac{1}{m}}]$ decrease to the empty set as $\delta \searrow 0$, then for each fixed N, the probability $\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\delta,\frac{1}{m}}]$ decreases to zero as $\delta \searrow 0$. Therefore, for each fixed $N \in \mathbb{N}$, we can choose

$$\tilde{\delta}_N = \tilde{\delta}_N(\ell, m, n) \leq \delta_0(\ell, m, n) = \delta_0 \tag{5.23}$$

such that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\tilde{\delta}_N,\frac{1}{m}}] \leq e^{-Nnm\ell}. \tag{5.24}$$

Denote now

$$\tilde{\delta} := \min_{N < N_0} \tilde{\delta}_N \leq \delta_0.$$

Let $N \in \mathbb{N}$. If $N < N_0$, then, by $\tilde{\delta} \leq \tilde{\delta}_N$, (5.21) and (5.24), we have that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\tilde{\delta},\frac{1}{m}}] \leq \mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\tilde{\delta}_N,\frac{1}{m}}] \leq e^{-Nnm\ell}.$$

Furthermore, if $N \geq N_0$, the construction $\tilde{\delta} \leq \delta_0$ [see (5.22) and (5.23)] assures that

$$\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell,\tilde{\delta},\frac{1}{m}}] \leq e^{-Nnm\ell},$$

completing the proof of the claim.

Keeping in mind that our goal is to prove that the sequence $\mathbb{Q}_{\mu_N}$ is exponentially tight, we define

$$K_n = \bigcap_{\ell \geq 1, m \geq 1} C_{\ell,\tilde{\delta},\frac{1}{m}},$$

which is a intersection of closed sets, hence closed as well. In order to prove that K_n is a compact set for each $n \geq 1$, we use a version of Arzelà–Ascoli theorem, which states that a set of functions $K_n \subset \mathcal{D}_{\mathcal{M}}$ is relatively compact if it is uniformly bounded, and

$$\lim_{\delta \to 0} \sup_{\pi \in K_n} \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} d(\pi_s, \pi_t) = 0, \tag{5.25}$$

where the infimum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_r$ with $t_i - t_{i-1} > \delta$ and d is the metric on $\mathcal{M}$. We start by observing that K_n is uniformly bounded, because $K_n \subset \mathcal{D}_{\mathcal{M}}$ [cf. the Definition of $\mathcal{M}$ in (2.3)]. The limit (5.25) is a consequence of

$$\lim_{\delta\to 0} \sup_{\pi\in K_n} \sup_{|t-s|\le\delta} d(\pi_s,\pi_t) = 0. \tag{5.26}$$

To prove the limit above, we start by observing that if $\pi \in K_n$ and $|t-s| \le \tilde{\delta}$ (we can suppose without loss of generality that $s \le t$, thus $s \le t \le s+\tilde{\delta}$), then

$$\bigl|\langle \pi_t, H_\ell\rangle - \langle \pi_s, H_\ell\rangle\bigr| \le \frac{1}{m}, \qquad \forall \ell, m \in \mathbb{N}.$$

Now, recalling that the metric d on $\mathcal{M}$ is

$$d(\pi_s,\pi_t) = \sum_{\ell\in\mathbb{N}} \frac{1}{2^\ell} \frac{|\langle \pi_t, H_\ell\rangle - \langle \pi_s, H_\ell\rangle|}{1+|\langle \pi_t, H_\ell\rangle - \langle \pi_s, H_\ell\rangle|} \le \sum_{\ell\in\mathbb{N}} \frac{1}{2^\ell}\bigl|\langle \pi_t, H_\ell\rangle - \langle \pi_s, H_\ell\rangle\bigr|,$$

we have, for $\pi \in K_n$ and $|t-s| \le \tilde{\delta}$, that $d(\pi_s,\pi_t) \le \frac{1}{m}$, for all $m \in \mathbb{N}$, leading to

$$\sup_{\pi\in K_n} \sup_{|t-s|\le\tilde{\delta}} d(\pi_s,\pi_t) \le \frac{1}{m} \qquad \text{for all } m \in \mathbb{N}. \tag{5.27}$$

Since $\delta \mapsto \sup_{|t-s|\le\delta} d(\pi_s,\pi_t)$ is decreasing on δ (for π fixed), the inequality (5.27) holds for $\delta \le \tilde{\delta}$ in place of $\tilde{\delta}$. Therefore, the limit (5.26) follows.

Since K_n is relatively compact and closed, we conclude that K_n is a compact set. Furthermore, by construction of the set K_n and the last claim, we have that

$$\mathbb{Q}_{\mu_N}[\pi \notin K_n] \le \sum_{\substack{\ell\ge 1\\ m\ge 1}} e^{-Nnm\ell} \le Ce^{-Nn},$$

where C is a constant not depending in the parameters. In particular,

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\pi \notin K_n] \le -n,$$

which is the exponential tightness. Therefore, it only remains to prove the Lemma 5.15.

Proof of Lemma 5.15. Fix $\varepsilon > 0$ and $H \in C^2(\mathbb{T})$. Recalling the definition of the set $\mathcal{C}_{H,\delta,\varepsilon}$, we can rewrite the set $[\pi \notin \mathcal{C}_{H,\delta,\varepsilon}]$ as

$$\Bigl\{\pi \in \mathcal{D}_{\mathcal{M}};\ \sup_{s\le t\le s+\delta} \bigl|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\bigr| > \varepsilon,\ \text{ for some } s \in [0,T]\Bigr\}.$$

Consider the partition of the interval $[0,T]$ with mesh size equal to δ. For each $s \in [0,T]$, there exists $k \in \{0,\dots,\lfloor T\delta^{-1}\rfloor\}$ such that $k\delta \le s < (k+1)\delta$. Thus,

$$\begin{aligned}
&\sup_{s\le t\le s+\delta} \bigl|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\bigr| \\
&\quad \le \sup_{s\le t\le (k+1)\delta} \bigl|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\bigr| + \sup_{(k+1)\delta\le t\le s+\delta} \bigl|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\bigr|.
\end{aligned}$$

Adding and subtracting $\langle \pi_{k\delta}, H\rangle$ in both terms above, and adding and subtracting $\langle \pi_{(k+1)\delta}, H\rangle$ in the second term, we bound the last expression by

$$4 \sup_{k\delta \le t \le (k+1)\delta} \big|\langle \pi_t, H\rangle - \langle \pi_{k\delta}, H\rangle\big| \\ + \sup_{(k+1)\delta \le t \le (k+2)\delta} \big|\langle \pi_t, H\rangle - \langle \pi_{(k+1)\delta}, H\rangle\big|.$$

Then

$$\Big\{\pi;\ \sup_{s\le t\le s+\delta} \big|\langle \pi_t, H\rangle - \langle \pi_s, H\rangle\big| > \varepsilon,\ \text{for some } s \in [0, T]\Big\} \subseteq \bigcup_{k=0}^{\lfloor T\delta^{-1}\rfloor} A^{H,N}_{k,\delta,\varepsilon},$$

where

$$A^{H,N}_{k,\delta,\varepsilon} = \Big\{\sup_{k\delta \le t \le (k+1)\delta} \big|\langle \pi_t, H\rangle - \langle \pi_{k\delta}, H\rangle\big| > \varepsilon/5\Big\}.$$

Thus, for all $\delta > 0$,

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\pi \notin \mathcal{C}_{\ell,\delta,\frac{1}{m}}] \le \varlimsup_{N\to\infty} \frac{1}{N} \log \sum_{k=0}^{\lfloor T\delta^{-1}\rfloor} \mathbb{Q}_{\mu_N}[A^{H,N}_{k,\delta,\varepsilon}]. \tag{5.28}$$

Since

$$\varlimsup_{N} N^{-1} \log\{a_N + b_N\} = \max\{\varlimsup_{N} N^{-1} \log a_N, \varlimsup_{N} N^{-1} \log b_N\}, \tag{5.29}$$

the limit in the right-hand side of (5.28) is bounded from above by

$$\max_{k\in\{0,\dots,\lfloor T\delta^{-1}\rfloor\}} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[A^{H,N}_{k,\delta,\varepsilon}].$$

Then, in order to prove the Lemma 5.15, it is enough to show that

$$\lim_{\delta\downarrow 0} \max_{k\in\{0,\dots,\lfloor T\delta^{-1}\rfloor\}} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[A^{H,N}_{k,\delta,\varepsilon}] = -\infty. \tag{5.30}$$

We begin by observing that $A^{H,N}_{k,\delta,\varepsilon} = B^{H,N}_{k,\delta,\varepsilon} \cup B^{-H,N}_{k,\delta,\varepsilon}$, where

$$B^{H,N}_{k,\delta,\varepsilon} = \Big\{\sup_{k\delta \le t \le (k+1)\delta} \langle \pi_t, H\rangle - \langle \pi_{k\delta}, H\rangle > \varepsilon/10\Big\}.$$

Hence, recalling (5.29), to obtain (5.30) it is sufficient to assure that

$$\lim_{\delta\downarrow 0} \max_{k\in\{0,\dots,\lfloor T\delta^{-1}\rfloor\}} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[B^{H,N}_{k,\delta,\varepsilon}] = -\infty, \tag{5.31}$$

for any $H \in C^2(\mathbb{T})$ and $\varepsilon > 0$. To obtain the claim above, we analyze the limit $\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[B^{H,N}_{k,\delta,\varepsilon}]$ for fixed k, δ, ε and H. Let $a > 0$. Denote

$$M^{a,H}_t = \exp\Big\{aN\Big[\langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t U^a_N(H, s, \eta_s)\, ds\Big]\Big\},$$

where

$$U_N^a(H,s,\eta_s) = \frac{1}{aN} e^{-aN\langle \pi_s^N, H\rangle}(\partial_s + N^2 L_N) e^{aN\langle \pi_s^N, H\rangle}.$$

Note that $M_t^{a,H}$ is a positive mean one martingale with respect to the natural filtration. And $\{M_t^{a,H}/M_{k\delta}^{a,H}\}_{t\geq k\delta}$ is also a positive mean one martingale. Adding and subtracting the integral part, we get

$$\mathbb{Q}_{\mu_N}[B_{k,\delta,\varepsilon}^{H,N}] \leq \mathbb{Q}_{\mu_N}[C_{k,\delta,\varepsilon}^{a,H,N}] + \mathbb{Q}_{\mu_N}[D_{k,\delta,\varepsilon}^{a,H,N}],$$

where

$$C_{k,\delta,\varepsilon}^{a,H,N} = \left\{ \sup_{k\delta\leq t\leq (k+1)\delta} \frac{1}{aN} \log\left(\frac{M_t^{a,H}}{M_{k\delta}^{a,H}}\right) > \varepsilon/20 \right\}$$

and

$$D_{k,\delta,\varepsilon}^{a,H,N} = \left\{ \sup_{k\delta\leq t\leq (k+1)\delta} \int_{k\delta}^t U_N^a(H,s,\eta_s)\,ds > \varepsilon/20 \right\}.$$

By the considerations above and again (5.29), we have that

$$\begin{aligned} &\varlimsup_{N\to\infty} \frac{1}{N}\log \mathbb{Q}_{\mu_N}[B_{k,\delta,\varepsilon}^{H,N}] \\ &\quad\leq \max\left\{ \varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[C_{k,\delta,\varepsilon}^{a,H,N}], \varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[D_{k,\delta,\varepsilon}^{a,H,N}] \right\}, \end{aligned} \tag{5.32}$$

for all $\delta > 0$ and $k \in \{0,1,\dots,\lfloor T\delta^{-1}\rfloor\}$. Since $H \in C^2(\mathbb{T})$, by Taylor expansion it is easy[5] to verify that $|\int_{k\delta}^t U_N^a(H,s,\eta_s)\,ds|$ is bounded by $C(a,H)\delta$, for all $t \in [k\delta,(k+1)\delta]$. Thus, if we take $\delta \in (0,\tilde{C})$ with $\tilde{C} := \varepsilon/(20C(a,H))$, then $\mathbb{Q}_{\mu_N}[D_{k,\delta,\varepsilon}^{a,H,N}] = 0$ for all $k \in \{0,1,\dots,\lfloor T\delta^{-1}\rfloor\}$ and, therefore, the inequality (5.32) becomes

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[B_{k,\delta,\varepsilon}^{H,N}] \leq \varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[C_{k,\delta,\varepsilon}^{a,H,N}],$$

provided $\delta < \tilde{C}$. We handle now the set $C_{k,\delta,\varepsilon}^{a,H,N}$ in the following way:

$$\mathbb{Q}_{\mu_N}[C_{k,\delta,\varepsilon}^{a,H,N}] = \mathbb{Q}_{\mu_N}\left[\sup_{k\delta\leq t\leq (k+1)\delta}\frac{M_t^{a,H}}{M_{k\delta}^{a,H}} > e^{aN\varepsilon/20}\right] \leq \frac{1}{e^{aN\varepsilon/20}},$$

where in last inequality we have used Doob's inequality since $\{M_t^{a,H}/M_{k\delta}^{a,H}\}_{t\geq k\delta}$ is a mean one positive martingale. Thus,

$$\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{\mu_N}[B_{k,\delta,\varepsilon}^{H,N}] \leq -a\varepsilon/20,$$

[5] One can do similar computations of those in the Section 4.2.

for all $a > 0$, $\varepsilon > 0$, $H \in C^2(\mathbb{T})$, $\delta \in (0, \tilde{C})$ and $k = 0, 1, \dots, \lfloor T\delta^{-1} \rfloor$. Fix $a > 0$. Taking the limit $\delta \searrow 0$ in the inequality above gives us

$$\varlimsup_{\delta \downarrow 0} \max_{k \in \{0,\dots,\lfloor T\delta^{-1}\rfloor\}} \varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[B^{H,N}_{k,\delta,\varepsilon}] \le \frac{-a\varepsilon}{20}.$$

Now, taking the limit when $a \to +\infty$ leads to (5.31), completing the proof. □

6. Large deviations lower bound for smooth profiles. Next, we obtain a nonvariational formulation of the rate functional I for profiles ρ whose are solutions of the hydrodynamical equation for some perturbation $H \in C^{1,2}([0,T] \times [0,1])$.

PROPOSITION 6.1. *Given $H \in C^{1,2}([0,T] \times [0,1])$, let ρ^H be the unique weak solution of* (2.10). *Then*

$$\begin{aligned} I(\rho^H) &:= \sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H) \\ &= \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t)^2 \rangle\, dt + \int_0^T \rho_t^H(0^-)(1-\rho_t^H(0^+))\Gamma(\delta H_t(0))\, dt \\ &\quad + \int_0^T \rho_t^H(0^+)(1-\rho_t^H(0^-))\Gamma(-\delta H_t(0))\, dt, \end{aligned} \tag{6.1}$$

where $\Gamma(y) = 1 - e^y + ye^y$, $\forall y \in \mathbb{R}$.

Although of quite simple proof, this result has a deep interpretation. The functional $-\hat{J}_G(\rho)$ has the meaning of being *the price* to observe the profile ρ when we perturb the system by G. The equality $\sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H)$ says that the minimum cost to observe the profile ρ is reached by picking up the perturbation $G = H$, where H is such that $\rho = \rho^H$, that is, such that ρ is a solution of (2.10).

PROOF OF PROPOSITION 6.1. Replacing the integral equation (2.12) in the definition of $\hat{J}$ given in (2.14), we get

$$\begin{aligned} \hat{J}_G(\rho^H) = {} & \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t)^2 \rangle\, dt - \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t - \partial_u G_t)^2 \rangle\, dt \\ & + \int_0^T \rho_t^H(0^-)(1-\rho_t^H(0^+))\bar{\Gamma}(\delta G_t(0), \delta H_t(0))\, dt \\ & + \int_0^T \rho_t^H(0^+)(1-\rho_t^H(0^-))\bar{\Gamma}(-\delta G_t(0), -\delta H_t(0))\, dt, \end{aligned}$$

where $\bar{\Gamma}(x,y) = 1 - e^x + xe^y$, $\forall x, y \in \mathbb{R}$. Let $y \in \mathbb{R}$ fixed. The function $x \mapsto \bar{\Gamma}(x,y)$ assumes its maximum at $x = y$. Therefore, $I(\rho^H) = \sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H)$. Noticing that $\Gamma(y) = \bar{\Gamma}(y,y)$ we arrive at (6.1). □

REMARK 6.2. As natural, if λ is the unique weak solution of (2.5), then the rate functional vanishes at λ. In fact, given $G \in C^{1,2}([0,T] \times [0,1])$, we have $\ell_G(\lambda) = 0$ because λ satisfies the integral equation (2.6). Since $\psi(u) = e^u - u - 1 \geq 0$, it yields $\hat{J}_G(\lambda) \leq 0$. And $\hat{J}_G(\lambda) = 0$ if G is constant.

By Proposition 6.1, profiles that are solution of (2.10) for some H provides a special representation for the rate functional. This motivates the next definition.

DEFINITION 6.3. Denote by $\mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}$ the subset of $\mathcal{D}_{\mathcal{M}_0}$ consisting of all paths $\pi_t(du) = \rho_t(u)\,du$ for which there exists some $H \in C^{1,2}([0,T] \times [0,1])$ such that $\rho = \rho^H$ is the unique weak solution of (2.10).

We begin by proving the lower bound for trajectories in $\mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}$. In the following, we present the lower bound in the set of smooth trajectories, $\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}$.

PROPOSITION 6.4. *Let $\mathcal{O}$ be an open set of $\mathcal{D}_{\mathcal{M}}$. Then*

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O} \cap \mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}} I(\pi).$$

PROOF. This proof is essentially the same as that found in [9]. Fix the open set $\mathcal{O}$. Given $\pi \in \mathcal{O} \cap \mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}$, by definition there exists $H \in C^{1,2}([0,T] \times [0,1])$ such that $\pi_t(du) = \rho^H_t(u)\,du$, where ρ^H is the weak solution of (2.10). Denote by $\mathbb{P}^{H,\mathcal{O}}_{\mu_N}$ the probability on the space $\mathcal{D}_{\Omega_N}$ defined by

$$\mathbb{P}^{H,\mathcal{O}}_{\mu_N}[A] = \frac{\mathbb{P}^H_{\mu_N}[A, \pi^N \in \mathcal{O}]}{\mathbb{P}^H_{\mu_N}[\pi^N \in \mathcal{O}]},$$

for any A measurable subset of $\mathcal{D}_{\Omega_N}$. Within this definition,

$$\frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] = \frac{1}{N} \log \mathbb{E}^{H,\mathcal{O}}_{\mu_N}\left[\frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}^H_{\mu_N}}\right] + \frac{1}{N} \log \mathbb{Q}^H_{\mu_N}[\mathcal{O}].$$

Since $\mathcal{O}$ is a open set that contains ρ^H, by the Proposition 4.1 the second term in the right-hand side of above converges to zero as N increases to infinity. Since the logarithm is a concave function, by Jensen's inequality the first term in the right-hand side of above is bounded from below by

$$\mathbb{E}^{H,\mathcal{O}}_{\mu_N}\left[\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}^H_{\mu_N}}\right].$$

Adding and subtracting the indicator function of the set $\{\pi^N \in \mathcal{O}^\complement\}$, the last expectation becomes

$$\frac{1}{\mathbb{Q}^H_{\mu_N}[\mathcal{O}]}\left\{-\frac{1}{N}\mathbf{H}(\mathbb{P}^H_{\mu_N} | \mathbb{P}_{\mu_N}) - \mathbb{E}^H_{\mu_N}\left[\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}^H_{\mu_N}} \mathbf{1}_{\{\pi^N \in \mathcal{O}^\complement\}}\right]\right\}, \tag{6.2}$$

where

$$\mathbf{H}(\mathbb{P}^H_{\mu_N}|\mathbb{P}_{\mu_N}) := \mathbb{E}^H_{\mu_N}\left[\log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}\right] = -\mathbb{E}^H_{\mu_N}\left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}^H_{\mu_N}}\right] \tag{6.3}$$

is the so-called relative entropy of $\mathbb{P}^H_{\mu_N}$ with respect to $\mathbb{P}_{\mu_N}$. Again by Proposition 4.1, we have that $\mathbb{Q}^H_{\mu_N}[\mathcal{O}]$ converges to one as N increases to infinity. By (4.4), the expression $\frac{1}{N}\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}^H_{\mu_N}}$ is bounded, hence the second term inside braces in (6.2) vanishes as N increases to ∞. Thus,

$$\varliminf_{N\to\infty} \frac{1}{N}\log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq \lim_{N\to\infty} -\frac{1}{N}\mathbf{H}(\mathbb{P}^H_{\mu_N}|\mathbb{P}_{\mu_N}) = -I(\rho^H),$$

where the last equality has an importance for itself and for this reason it is postponed to the Lemma 6.5 proved next. $\square$

LEMMA 6.5. *Let $H \in C^{1,2}([0,T]\times[0,1])$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\mathbf{H}(\mathbb{P}^H_{\mu_N}|\mathbb{P}_{\mu_N}) = I(\rho^H),$$

where ρ^H is the unique weak solution of (2.10).

PROOF. Using formula (6.3) for the relative entropy, we get

$$\frac{1}{N}\mathbf{H}(\mathbb{P}^H_{\mu_N}|\mathbb{P}_{\mu_N}) = \frac{1}{N}\mathbb{E}^H_{\mu_N}\left[\log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}\mathbf{1}_{B^H_{\delta,\varepsilon}}\right] + \frac{1}{N}\mathbb{E}^H_{\mu_N}\left[\log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}\mathbf{1}_{(B^H_{\delta,\varepsilon})^\complement}\right], \tag{6.4}$$

where the set $B^H_{\delta,\varepsilon}$ was defined in (5.3). We claim that the event $(B^H_{\delta,\varepsilon})^\complement$ is super-exponentially small with respect to $\mathbb{P}^H_{\mu_N}$. Indeed, by (4.4) we have

$$\mathbb{P}^H_{\mu_N}[(B^H_{\delta,\varepsilon})^\complement] = \mathbb{E}_{\mu_N}\left[\frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}\mathbf{1}_{(B^H_{\delta,\varepsilon})^\complement}\right] \leq e^{C(H,T)N}\mathbb{P}_{\nu^N_\alpha}[(B^H_{\delta,\varepsilon})^\complement]$$

and then by (5.4) we get

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}^H_{\mu_N}[(B^H_{\delta,\varepsilon})^\complement] = -\infty.$$

Provided by the limit above and the fact that $\frac{1}{N}\log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}$ is bounded, the right-hand side of (6.4) is

$$\frac{1}{N}\mathbb{E}^H_{\mu_N}\left[\log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}\mathbf{1}_{B^H_{\delta,\varepsilon}}\right] + o_N(1), \tag{6.5}$$

for all $\delta > 0$ and each small enough $\varepsilon = \varepsilon(\delta)$. Applying the expression (5.9) for the Radon–Nikodym derivative, $\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}^H_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}}$ on the set $B^H_{\delta,\varepsilon}$ is equal to

$$\hat{J}_H\big((\pi^N * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big) + O_{H,T,\varepsilon,\gamma}\left(\frac{1}{N}\right) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right),$$

for all $\delta > 0$ and all ε and γ small enough. Since this expression is bounded and the probability of $(B^H_{\delta,\varepsilon})^{\complement}$ with respect to $\mathbb{P}^H_{\nu^N_\alpha}$ vanishes as N increases to infinity, the expression (6.5) becomes

$$\mathbb{E}^H_{\mu_N}\big[\hat{J}_H\big((\pi^N * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big)\big] + O_{H,T,\varepsilon,\gamma}\left(\frac{1}{N}\right) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right) + o_N(1),$$

for all $\delta > 0$ and all ε and γ small enough. For fixed ε and γ, the map $\rho \mapsto \hat{J}_H((\rho * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon)$ is continuous with respect to the Skorohod topology; see the Proposition 5.11. Moreover, by Proposition 4.1 the sequence $\mathbb{Q}^H_{\mu_N}$ converges weakly to the probability concentrated on the weak solution of (2.10). In particular, as N increases to infinity, the previous expectation converges to

$$\hat{J}_H\big((\rho^H * \iota^{\mathrm{s}}_\gamma) * \iota_\varepsilon\big) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right).$$

Letting $\gamma \downarrow 0$, then taking $\varepsilon \downarrow 0$, finally $\delta \downarrow 0$ and then invoking Lemma 6.1 concludes the proof. $\square$

Since weak solutions of (2.10) for some H implies the special representation (6.1) for the rate functional, it is natural to study in what conditions a profile ρ can be written as a solution of (2.10). This is the content of the next proposition. Notice that the first equation in (6.6) ahead is nothing else than the partial differential equation (2.10) rearranged.

PROPOSITION 6.6. *Let $\rho \in C^{1,2}([0,T] \times [0,1])$ such that $0 < \varepsilon \leq \rho \leq 1 - \varepsilon$, for some $\varepsilon > 0$. Then there exists a unique (strong) solution $H \in C^{1,2}([0,T] \times [0,1])$ of the elliptic equation*

$$\begin{cases} \partial_u^2 H_t(u) + \dfrac{\partial_u(\chi(\rho_t(u)))}{\chi(\rho_t(u))}\partial_u H_t(u) = \dfrac{\Delta\rho_t(u) - \partial_t\rho_t(u)}{2\chi(\rho_t(u))} & \forall u \in (0,1), \\ \partial_u H_t(0) = \dfrac{1}{2\chi(\rho_t(0))}\big[Be^{\delta H_t(0)} - Ce^{-\delta H_t(0)} + \partial_u\rho_t(0)\big], \\ \partial_u H_t(1) = \dfrac{1}{2\chi(\rho_t(1))}\big[Be^{\delta H_t(0)} - Ce^{-\delta H_t(0)} + \partial_u\rho_t(1)\big], \\ H_t(0) = 0, \end{cases} \tag{6.6}$$

where $B = B(\rho_t) = \rho_t(1)(1 - \rho_t(0))$ and $C = C(\rho_t) = \rho_t(0)(1 - \rho_t(1))$, for all $t \in [0,T]$. Above we are denoting $0 = 0^+$ and $1 = 0^-$.

PROOF. For fixed time, the first equation in (6.6) is a linear second-order ordinary differential equation in H. The only work is to adjust the solution to satisfy the boundary conditions. Let $z_0 \in \mathbb{R}$ be the unique solution of the transcendental equation $z = (Be^{-z} - Ce^{z})\alpha + A$, where

$$\alpha = \alpha(\rho_t) := \int_0^1 \frac{1}{2\chi(\rho_t(v))}\, dv,$$

$$A = A(\rho_t) := \int_0^1 \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\, dw}{2\chi(\rho_t(v))}\, dv,$$

and $B > 0$ and $C > 0$ are those ones in the statement of the proposition. Let

$$H_t(u) := \big(Be^{-z_0} - Ce^{z_0}\big) \int_0^u \frac{1}{2\chi(\rho_t(v))}\, dv + \int_0^u \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\, dw}{2\chi(\rho_t(v))}\, dv,$$

for all $t \in [0, T]$. It can be directly checked that H is the solution of (6.6). $\square$

Recalling the definition of $\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}$ given in the Theorem 2.12 and the definition of $\mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}$, Proposition 6.6 can be resumed as the following.

COROLLARY 6.7. *The set*

$$\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0} \cap \big\{\pi \in \mathcal{D}_{\mathcal{M}}; \pi_t(du) = \rho_t(u)\, du,\ \text{with } \varepsilon \le \rho \le 1 - \varepsilon \text{ for some } \varepsilon > 0\big\}$$

is contained in $\mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}$.

The next proposition shows that the rate functional I obtained in our model is convex under certain conditions.

PROPOSITION 6.8. *Let* $\rho, \lambda \in \mathcal{D}_{\mathcal{M}}$ *with* $I(\rho)$ *and* $I(\lambda)$ *finite such that* $(\rho_t(0^+) - \lambda_t(0^+))(\rho_t(0^-) - \lambda_t(0^-)) \ge 0$, *almost surely in* $t \in [0, T]$. *Then*, *for* $\theta \in [0, 1]$,

$$I\big(\theta\rho + (1-\theta)\lambda\big) \le \theta I(\rho) + (1-\theta) I(\lambda). \tag{6.7}$$

PROOF. Let $\theta \in [0, 1]$. We claim that

$$\hat{J}_H\big(\theta\rho + (1-\theta)\lambda\big) \le \theta \hat{J}_H(\rho) + (1-\theta)\hat{J}_H(\lambda), \tag{6.8}$$

for any $H \in C^{1,2}([0, T] \times [0, 1])$. Recall that $\hat{J}_H(\rho)$ is the sum of linear part in ρ, namely

$$\ell_H(\rho) - \int_0^T \big\{\rho_t(0^-)\psi\big(\delta H_t(0)\big) + \rho_t(0^+)\psi\big(-\delta H_t(0)\big)\big\}\, dt,$$

plus a convex part in ρ, namely $-\int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2 \rangle\, dt$, and

$$\int_0^T \rho_t(0^-)\rho_t(0^+)\big\{\psi\big(\delta H_t(0)\big) + \psi\big(-\delta H_t(0)\big)\big\}\, dt, \tag{6.9}$$

wherefore we only need to care about this last term. Since $\psi(x) = e^x - x - 1 \geq 0$, we have that $\psi(\delta H_t(0)) + \psi(-\delta H_t(0)) \geq 0$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = xy$. If (x_1, y_1) and (x_2, y_2) are two points of $\mathbb{R}^2$ such that $(x_2 - x_1)(y_2 - y_1) \geq 0$, then

$$f\big(\theta(x_1, y_1) + (1-\theta)(x_2, y_2)\big) \leq \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2). \tag{6.10}$$

To see this, just note that f is convex along lines of the form $y = ax + b$, provided $a > 0$. Inequality (6.10) applied to (6.9) permits us to conclude inequality (6.8), which leads to (6.7). □

PROPOSITION 6.9. *Let $\pi \in \mathcal{D}_{\mathcal{M}}$ with $I(\pi) < \infty$. There exists a sequence $\{\pi^\varepsilon\}_{\varepsilon>0}$ in $\mathcal{D}_{\mathcal{M}_0}$ such that π^ε converges to π in $\mathcal{D}_{\mathcal{M}}$ and $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u)\, du$ with $\varepsilon \leq \rho_t^\varepsilon(u) \leq 1 - \varepsilon$. Moreover, $\overline{\lim}_{\varepsilon\downarrow 0} I(\pi^\varepsilon) \leq I(\pi)$.*

PROOF. Let $\pi \in \mathcal{D}_{\mathcal{M}}$ with $I(\pi) < \infty$, then $\pi_t(du) = \rho_t(u)\, du$ and $0 \leq \rho \leq 1$. Consider $\tilde{1}(t, u) = 1$ and $\tilde{0}(t, u) = 0$, for all $t \in [0, T]$ and $u \in \mathbb{T}$. Define $\rho^\varepsilon = \varepsilon\tilde{1} + (1 - 2\varepsilon)\rho + \varepsilon\tilde{0}$ and $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u)\, du$. By Lemma 6.8, $I(\pi^\varepsilon) \leq \varepsilon I(\tilde{1}) + (1 - 2\varepsilon) I(\rho) + \varepsilon I(\tilde{0})$. Hence, $\overline{\lim}_{\varepsilon\downarrow 0} I(\pi^\varepsilon) \leq I(\pi)$. □

We are in position to prove the lower bound for smooth profiles.

PROOF OF THEOREM 2.12(ii). Fix $\pi \in \mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0} \cap \mathcal{O}$ and consider the sequence $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u)\, du$, where $\rho_t^\varepsilon(u) = \varepsilon + (1-2\varepsilon)\rho_t(u)$, as in the proof of the Proposition 6.9. That is, such that $\varepsilon < \rho^\varepsilon < 1 - \varepsilon$ with $\rho^\varepsilon \in C^{1,2}([0, T] \times [0, 1])$. By Corollary 6.7 and since $\mathcal{O}$ is open, we have that $\pi^\varepsilon \in \mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0} \cap \mathcal{O}$ for small enough $\varepsilon > 0$.

By Proposition 6.4,

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq - \inf_{\lambda \in \mathcal{O}\cap\mathcal{D}^{\mathrm{eq}}_{\mathcal{M}_0}} I(\lambda) \geq -I\big(\pi^\varepsilon\big).$$

Taking the limit infimum in the right-hand side of inequality above and using the Lemma 6.9, we get

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq - \varlimsup_{\varepsilon\to 0} I\big(\pi^\varepsilon\big) \geq -I(\pi).$$

Since π is an arbitrary trajectory on the set $\mathcal{O} \cap \mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}$, we can optimize over all elements in this set, obtaining therefore

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq \sup_{\pi \in \mathcal{O}\cap\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}} -I(\pi) = - \inf_{\pi \in \mathcal{O}\cap\mathcal{D}^{\mathcal{S}}_{\mathcal{M}_0}} I(\pi),$$

which completes the proof. □

REFERENCES

[1] BERTINI, L., LANDIM, C. and MOURRAGUI, M. (2009). Dynamical large deviations for the boundary driven weakly asymmetric exclusion process. *Ann. Probab.* **37** 2357–2403. MR2573561

[2] FAGGIONATO, A., JARA, M. and LANDIM, C. (2009). Hydrodynamic behavior of 1D subdiffusive exclusion processes with random conductances. *Probab. Theory Related Fields* **144** 633–667. MR2496445

[3] FARFAN, J., LANDIM, C. and MOURRAGUI, M. (2011). Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.* **121** 725–758. MR2770905

[4] FRANCO, T., GONÇALVES, P. and NEUMANN, A. (2013). Hydrodynamical behavior of symmetric exclusion with slow bonds. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 402–427. MR3088375

[5] FRANCO, T., GONÇALVES, P. and NEUMANN, A. (2013). Phase transition in equilibrium fluctuations of symmetric slowed exclusion. *Stochastic Process. Appl.* **123** 4156–4185. MR3096351

[6] FRANCO, T., GONÇALVES, P. and NEUMANN, A. (2015). Phase transition of a heat equation with Robin's boundary conditions and exclusion process. *Trans. Amer. Math. Soc.* **367** 6131–6158. MR3356932

[7] FRANCO, T. and LANDIM, C. (2010). Hydrodynamic limit of gradient exclusion processes with conductances. *Arch. Ration. Mech. Anal.* **195** 409–439. MR2592282

[8] FRANCO, T. and NEUMANN, A. (2015). Large deviations for the exclusion process with a slow bond. Available at arXiv:1501.00225v1.

[9] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314

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