Probabilidade 2017.1 Prova 2 Gabarito Resumido

(1) $\mathbb{E}X = \int_0^{+\infty}(1-F(x))dx - \int_{-\infty}^{0} F(x)dx = \int_0^1 (1-F(x))dx - \int_{-2}^{0} F(x)dx$

$= \int_0^1 \left(1 - \frac{x}{10} - \frac{3}{5}\right)dx - \int_{-2}^{0}\left(\frac{x}{10} + \frac{3}{5}\right)dx = \left(\frac{2}{5}x - \frac{x^2}{20}\right)\Big|_0^1 - \left(\frac{x^2}{20} + \frac{3x}{5}\right)\Big|_{-2}^{0}$

$= \left(\frac{2}{5} - \frac{1}{20}\right) - \left(\frac{4}{20} - \frac{6}{5}\right) = \frac{8}{5} - \frac{3}{20} = \frac{32-3}{20} = \frac{29}{20}$

(2) $\mathbb{E}XY = \int_0^1\int_{-1}^{1-x} xy\cdot\frac{2}{3}\,dy\,dx = \int_0^1 \frac{xy^2}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\Big|_{-1}^{1-x} dx =$

$= \int_0^1\left(\frac{x(1-x)^2}{3} - \frac{x}{3}\right)dx = \int_0^1 \frac{\cancel{x}(1-2x+x^2) - \cancel{x}}{3}dx = \int_0^1 \frac{x^3-2x^2}{3}dx$

$= \frac{x^4}{3\cdot 4} - \frac{2x^3}{3\cdot 3}\Big|_0^1 = \frac{1}{12} - \frac{2}{9} = \frac{3-8}{36} = -\frac{5}{36}$

$\mathbb{E}X = \int_0^1\int_{-1}^{1-x} x\,\frac{2}{3}\,dy\,dx = \int_0^1 \frac{2xy}{3}\Big|_{-1}^{1-x} dx = \int_0^1 \frac{2x(1-x)+2x}{3}dx$

$= \frac{1}{3}\int_0^1 (4x - 2x^2)dx = \frac{1}{3}\left(\frac{4x^2}{2} - \frac{2x^3}{3}\right)\Big|_0^1 = \frac{1}{3}\left(2 - \frac{2}{3}\right) = \frac{4}{9}$

$\mathbb{E}Y = \int_0^1\int_{-1}^{1-x} y\,\frac{2}{3}\,dy\,dx = \int_0^1 \frac{y^2}{3}\Big|_{-1}^{1-x} dx = \int_0^1 \frac{(1-x)^2-1}{3}dx =$

$= \int_0^1 \frac{\cancel{1} - 2x + x^2 - \cancel{1}}{3}dx = \int_0^1 \frac{x^2-2x}{3}dx = \frac{x^3}{9} - \frac{2x^2}{6}\Big|_0^1 = \frac{1}{9} - \frac{1}{3} = \frac{-2}{9}$

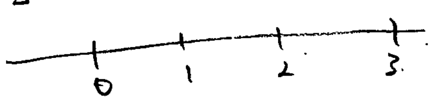
$\mathrm{cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y = -\frac{5}{36} - \frac{4}{9}\cdot\frac{(-2)}{9} = -\frac{5}{36} + \frac{8}{81} = \frac{-45+32}{324} = \frac{-13}{324}$

(3) $\mathbb{P}[A \mid Y=y] = \lim_{t\to 0} \mathbb{P}[A \mid Y \in (y-t, y+t)]$. Logo,

$\mathbb{P}[X \le 1 \mid Y = y] = \lim_{t\to 0} \mathbb{P}[X \le 1 \mid \max\{X,2\} \in (y-t, y+t)]$

$= \lim_{t\to 0} \frac{\mathbb{P}[X \le 1,\ \max\{X,2\} \in (y-t,y+t)]}{\mathbb{P}[\max\{X,2\} \in (y-t,y+t)]} = \circledast$

Temos que $\max\{X,2\} \in [2,3]$. Temos dois casos, $y=2$ e $y \in (2,3]$. Se $y \in (2,3]$,



então $[X \le 1,\ \max\{X,2\} \in (y-t, y+t)] = \emptyset$ e

$\mathbb{P}[X \le 1 \mid Y=y] = 0.$

Se $y=2$, $[X\leq 1, \max\{X,2\} \in (y-\epsilon, y+\epsilon)] = [X \leq 1]$, e

$$\mathbb{P}[X \leq 1 \mid Y = y] = \frac{1}{2}$$

④ $A_n =$ [todos os dados da n-ésima rodada têm mesmo resultado]

$$\mathbb{P}[A_n] = 6 \cdot \left(\frac{1}{6}\right)^n = \frac{1}{6^{n-1}} \cdot \text{ logo } \sum_{n=1}^{+\infty} \mathbb{P}(A_n) < +\infty \Rightarrow$$

$\mathbb{P}[A_n \text{ i.v.}] = 0$ por Borel-Cantelli.

⑤ X_k são independentes. Além disso, $\mathbb{E}X_k = 0$ e

$$\operatorname{Var} X_k = \mathbb{E}X^2 = \int_{-k^\theta}^{k^\theta} x^2 \frac{dx}{2k^\theta} = \frac{1}{2k^\theta} \cdot \frac{x^3}{3}\Big|_{-k^\theta}^{k^\theta} = \frac{k^{3\theta}}{3k^\theta} = \frac{k^{2\theta}}{3}.$$

Portanto,

$$\sum_{k=1}^{+\infty} \frac{\operatorname{Var} X_k}{k^2} = \sum_{k=1}^{+\infty} \frac{k^{2\theta}}{3k^2} = \frac{1}{3}\sum_{k=1}^{+\infty} \frac{1}{k^{2-2\theta}} < +\infty$$

pois $2 - 2\theta > 1$. Assim, pela 1ª Lei Forte de Kolmogorov,

$$\frac{S_n - \mathbb{E}S_n}{n} \longrightarrow 0 \text{ q.c.} \Rightarrow \frac{S_n}{n} \longrightarrow 0 \text{ q.c.}$$