Probabilidade Prova 2

Semestre 2016.1

Gabarito resumido

① $\operatorname{cov}(X, \mathbb{E}[Y|X]) = \mathbb{E}[X\,\mathbb{E}[Y|X]] - \mathbb{E}X \cdot \mathbb{E}[\mathbb{E}[Y|X]]$

$$= \mathbb{E}[\mathbb{E}[XY|X]] - \mathbb{E}X \cdot \mathbb{E}Y$$

$$= \mathbb{E}XY - \mathbb{E}X \cdot \mathbb{E}Y = \operatorname{cov}(X,Y)$$

② $\displaystyle\int_{-\infty}^{+\infty} x\,dF(x) = \frac{1}{2}\left(1-\frac{1}{2^\alpha}\right)\cdot 0 + \frac{1}{2}\left(1-\frac{1}{2^\alpha}\right)\cdot 2$

$$+ \int_2^{+\infty} x \frac{\alpha}{x^{\alpha+1}}\,dx$$

$$= \left(1-\frac{1}{2^\alpha}\right) + \int_2^{+\infty} \frac{\alpha}{x^\alpha}\,dx = \begin{cases} +\infty, & \text{se } \alpha \le 1 \\ \left(1-\frac{1}{2^\alpha}\right) + \left.\dfrac{\alpha x^{-\alpha+1}}{-\alpha+1}\right|_2^{+\infty}, & \alpha > 1 \end{cases}$$

$$= \begin{cases} +\infty, & \alpha \le 1 \\ \left(1-\frac{1}{2^\alpha}\right) + \dfrac{\alpha\, 2^{1-\alpha}}{(\alpha-1)}, & \alpha > 1 \end{cases}$$

③ $\mathbb{E}[\cos(XY)|Y] = g(Y)$, onde

$$g(y) = \mathbb{E}[\cos(Xy)] = \int_0^1 \cos(xy)\,dx = \left.\frac{\operatorname{sen} xy}{y}\right|_0^1$$

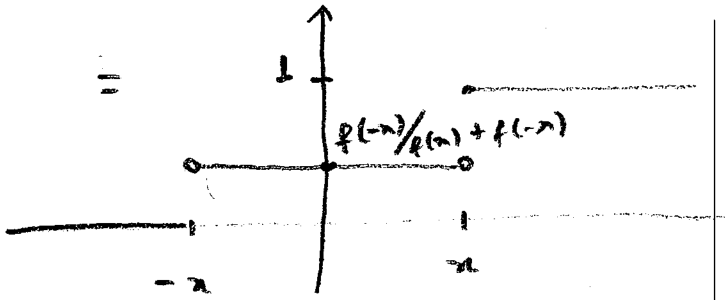
$$= \frac{\operatorname{sen} y}{y}$$

logo, $\mathbb{E}[\cos(XY)|Y] = \dfrac{\operatorname{sen} Y}{Y}$

(4) $\mathbb{P}[X \le x \mid |X|] = F(|X|)$, onde

$$F(x) = \lim_{\varepsilon \downarrow 0} \mathbb{P}[X \le x \mid |X| \in (x-\varepsilon, x+\varepsilon)]$$

$=$



(5) $$\mathbb{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{+\infty} \frac{1}{x}\, dF(x) = \frac{1}{2}\cdot\left(1-\frac{1}{2^\alpha}\right) + \int_2^{+\infty} \frac{1}{x}\frac{\alpha}{x^{\alpha+1}}\, dx$$

$$= \frac{1}{2}\left(1-\frac{1}{2^\alpha}\right) + \int_2^{+\infty} \frac{\alpha}{x^{\alpha+2}}\, dx$$

$$= \frac{1}{2}\left(1-\frac{1}{2^\alpha}\right) + \alpha \left.\frac{x^{-(\alpha+2)+1}}{-(\alpha+2)+1}\right|_2^{+\infty}$$

$$= \frac{1}{2}\left(1-\frac{1}{2^\alpha}\right) + \alpha \left.\frac{x^{-\alpha-1}}{-\alpha-1}\right|_2^{+\infty} < +\infty \text{ para } \alpha > 0$$

$\therefore$ aplique LGN.

2ª Parte

1) a) $\sum_{n=1}^{+\infty} \mathbb{P}[X_n = 1] = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} = +\infty.$

Como são indep., por Borel-Cantelli $\mathbb{P}[X_n = 1 \text{ i.v}] = 1.$

b) $A_{2n} = [X_{2n} = 1] \cap [X_{2n+1} = 1]$

A_{2n} são independentes.

$$\sum_{n=1}^{+\infty} \mathbb{P}(A_{2n}) = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{2n(2n+1)}} = +\infty \overset{\text{Borel-Cantelli}}{\Rightarrow} \mathbb{P}[A_{2n} \text{ i.v}] = 1.$$

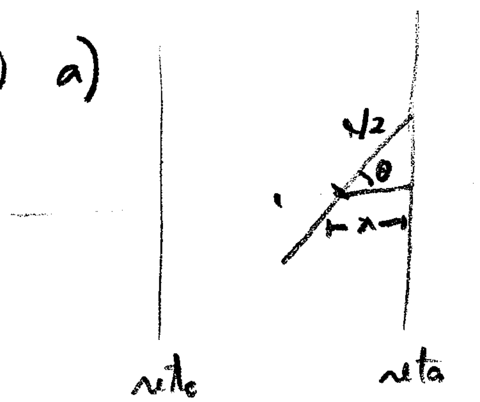
Como $[[X_n = 1] \cap [X_{n+1} = 1] \text{ i.v}] \supseteq [A_{2n} \text{ i.v}]$

$\Rightarrow \mathbb{P}[[X_n = 1] \cap [X_{n+1} = 1] \text{ i.v.}] = 1.$

c) $\mathbb{P}[X_n = 1, X_{n+1} = 1, X_{n+2} = 1] = \frac{1}{\sqrt{n(n+1)(n+2)}}$

$\overset{\text{Borel-Cantelli}}{\Rightarrow} \mathbb{P}[[X_n = 1] \cap [X_{n+1} = 1] \cap [X_{n+2} = 1] \text{ i.v}] = 0.$

2) a)



$$P = \int_0^{\pi/2} \frac{2}{\pi} \int_0^{\frac{l\cos\theta}{2}} dx \cdot \frac{2}{d}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{l\cos\theta}{2} \cdot \frac{2}{d} \cdot \frac{2l}{2\pi} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$\boxed{\begin{array}{l} \frac{x}{l/2} = \cos\theta \\ x = \frac{l\cos\theta}{2} \end{array}}$$

$$= \frac{2l}{2\pi} \cdot \operatorname{sen}\theta \Big|_0^{\pi/2} \cdot \frac{2}{d}$$

$$= \frac{2l}{2\pi} \cdot \frac{2}{d\pi} = \frac{2l}{d\pi}$$

2) b) Lei dos Grandes Números

3) $\mathbb{P}[S_n \geq \varepsilon n] = \mathbb{P}[\lambda S_n \geq \lambda \varepsilon n]$

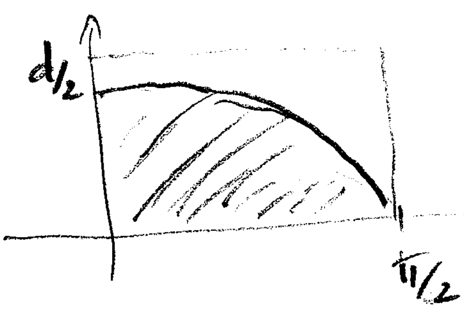
$$= \mathbb{P}[e^{\lambda S_n} \geq e^{\lambda \varepsilon n}] \overset{\text{Chebyshev}}{\leq} \frac{\mathbb{E}[e^{\lambda S_n}]}{e^{\lambda \varepsilon n}}$$

$$\Rightarrow \frac{1}{n} \log \mathbb{P}[S_n \geq \varepsilon n] \leq -\lambda\varepsilon + \frac{1}{n} \log \mathbb{E}[e^{\lambda S_n}]$$

$$= -\lambda\varepsilon + \log \mathbb{E}[e^{\lambda X_1}]$$

4) Use $\varepsilon = \mathbb{E}X_1 + \delta$, δ pequeno e minimize em λ do lado direito.

5) Sejam $X_n \sim U[0, \frac{d}{2}]$, $Y_n \sim U[0, \pi/2]$ todas independentes (o computador tem como gerar $U[0,1]$) sorteie (X_1, Y_1) e conte 1 se $(X_1, Y_1) \in R$, com R:



Repita o algoritmo n vezes e tome a média, etc.