Probabilidade Prova 1

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Gabarito Resumido

1) $$\mathbb{P}\left[\bigcap_{k=1}^{+\infty} A_k\right] = 1 - \mathbb{P}\left[\left(\bigcap_{k=1}^{+\infty} A_k\right)^c\right] = 1 - \mathbb{P}\left[\bigcup_{k=1}^{+\infty} A_k^c\right]$$

$$\geq 1 - \sum_{k=1}^{+\infty} \mathbb{P}(A_k^c)$$

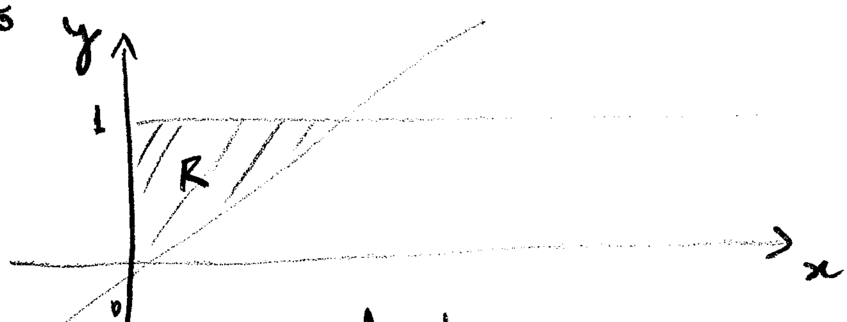
2) $$\mathbb{P}(A \cap B^c) = \mathbb{P}(A - A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$$

$$\overset{\text{indep.}}{=} \mathbb{P}(A) - \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A)\,(1 - \mathbb{P}(B))$$

$$= \mathbb{P}(A)\,\mathbb{P}(B^c).$$

Logo, A e B^c são independentes. Trocando A por B^c e B^c por A e usando o resultado acima concluímos que A^c e B^c são independentes.

3) $\mathbb{P}[X < Y] = \mathbb{P}[(X,Y) \in R]$, onde R é a região



$$\mathbb{P}[(X,Y) \in R] = \int_0^1 \int_0^y \lambda e^{-\lambda x}\, dx\, dy = \frac{\lambda + e^{-\lambda} - 1}{\lambda}$$

②

4) $\mathbb{P}[X+Y=k] = \mathbb{P}\left[\bigcup_{j=0}^{k} [X=j, Y=k-j]\right]$

$$= \sum_{j=0}^{k} \mathbb{P}[X=j]\cdot\mathbb{P}[Y=k-j] = \sum_{j=0}^{k} e^{-\lambda_1}\frac{\lambda_1^j}{j!}\cdot e^{-\lambda_2}\frac{\lambda_2^{k-j}}{(k-j)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\lambda_1^j\lambda_2^{k-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k$$

5) $\mathbb{E}[Y\mathbb{1}_{[Y>0]}] \overset{\text{Cauchy-Schwarz}}{\leq} \sqrt{\mathbb{E}Y^2\cdot\mathbb{E}[\mathbb{1}^2_{[Y>0]}]}$

$$= \sqrt{\mathbb{E}Y^2\cdot\mathbb{P}[Y>0]}$$

Como $Y\mathbb{1}_{[Y>0]} \geq Y \Rightarrow \mathbb{E}[Y\mathbb{1}_{[Y>0]}] \geq \mathbb{E}Y$

$$\Rightarrow \mathbb{E}Y \leq \sqrt{\mathbb{E}Y^2\cdot\mathbb{P}[Y>0]}$$

$$\Rightarrow \mathbb{P}[Y>0] \geq \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2}$$

b) $\mathbb{E}[h(X,Y)] = \iint h(x,y)\, f(x,y)\, dx\, dy$. Dai,

$$\mathbb{E}[XY] = \int_0^4 \int_0^{1-\frac{x}{4}} \frac{1}{2} xy \, dy\, dx = \int_0^4 \frac{1}{2} \frac{xy^2}{2} \Big|_0^{1-\frac{x}{4}} dx$$

$$= \frac{1}{2}\int_0^4 \frac{x(1-x/4)^2}{2}\, dx = \frac{1}{2}\int_0^4 \frac{x}{2}\left(1 - \frac{x}{2} + \frac{x^2}{16}\right) dx$$

$$= \frac{1}{2}\int_0^4 \left(\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{32}\right) dx = \frac{1}{2}\left(\frac{x^2}{4} - \frac{x^3}{12} + \frac{x^4}{128} \Big|_0^4\right)$$

$$= \frac{1}{2}\left(\frac{16}{4} - \frac{4^3}{12} + \frac{4^4}{128}\right) = \frac{1}{2}\left(4 - \frac{16}{3} + 4\right) = \frac{4}{3}$$

$$\mathbb{E}[X] = \frac{1}{2}\int_0^4 \int_0^{1-x/4} x\, dy\, dx = \frac{1}{2}\int_0^4 xy \Big|_0^{1-x/4} dx$$

$$= \frac{1}{2}\int_0^4 x\left(1 - \frac{x}{4}\right) dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{12} \Big|_0^4\right) = \frac{1}{2}\left(8 - \frac{4^2}{3}\right) = \frac{4}{3}$$

$$\mathbb{E}[Y] = \frac{1}{2}\int_0^4 \int_0^{1-\frac{x}{4}} y\, dy\, dx = \frac{1}{2}\int_0^4 \frac{y^2}{2} \Big|_0^{1-\frac{x}{4}} dx$$

$$= \frac{1}{4}\int_0^4 (1 - x/4)^2\, dx = \frac{1}{4}\int_0^4 \left(1 - \frac{x}{2} + \frac{x^2}{16}\right) dx$$

$$= \frac{1}{4}\left(x - \frac{x^2}{4} + \frac{x^3}{48}\right)\Big|_0^4 = \frac{1}{4}\left(\cancel{4} - \cancel{\frac{16}{4}} + \frac{4^3}{48}\right) =$$

$$= \frac{1}{3}$$

$\mathrm{Cor}(X,Y) = 8/9$

7) $\mathbb{E}(X-c)^2 = \mathbb{E}(X-\mu+\mu-c)^2$

$$= \mathbb{E}(X-\mu)^2 + 2\,\mathbb{E}(X-\mu)(\mu-c) + (c-\mu)^2$$

$$= \mathbb{E}(X-\mu)^2 + (c-\mu)^2 \geq \mathbb{E}(X-\mu)^2,$$

valendo a igualdade apenas quando $\mu = c$.

Logo, $\mathbb{E}(X-\mu)^2 = \min\limits_{c} \mathbb{E}(X-c)^2$

Seja $a \leq X \leq b$. Escolhendo $c = \frac{b+a}{2}$,

$$\mathrm{Var}(X) \leq \mathbb{E}\left[\left(X - \frac{b+a}{2}\right)^2\right]$$

$$\leq \mathbb{E}\left[\left(\frac{b-a}{2}\right)^2\right] = \frac{(b-a)^2}{4}$$