Teoria Espectral 2015.2

Prova 2 04/05/15

Gabarito Resumido

1) a) Integração por partes

b) $$\int_{\mathbb{R}} |f(x)|^2 dx \leq 2 \int_{\mathbb{R}} \left| x \, \mathrm{Re}\left(f(x) f'(x) \right) \right| dx$$

$$\leq 2 \int_{\mathbb{R}} |x f(x)| \cdot |f'(x)| \, dx \overset{\text{Hölder}}{\leq} 2 \, \|x f(x)\|_2 \, \|f'(x)\|_2$$

$$\overset{\text{Plancherel}}{=} 2 \|x f(x)\|_2 \, \|\widehat{f'(x)}\|_2$$

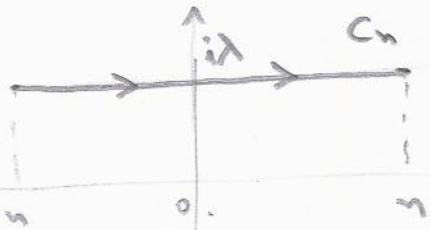
$$= 2 \|x f(x)\|_2 \cdot \| i\lambda f(\lambda) \|_2$$

$$= 2 \|x f(x)\|_2 \cdot \| \lambda f(\lambda) \|_2$$

2) a) $$\mathcal{F}\left[e^{-x^2/2} \right](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\lambda} e^{-x^2/2} dx$$

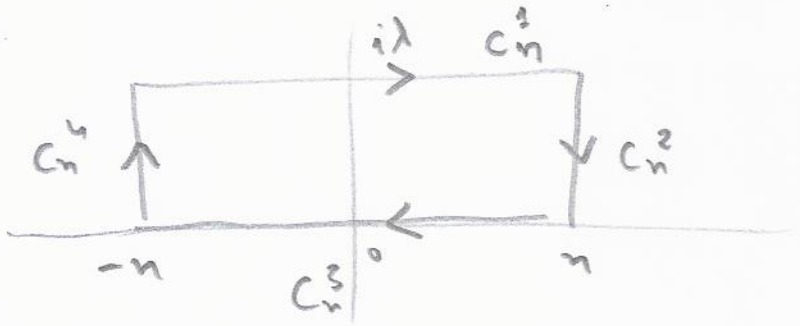
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[x^2 + 2i\lambda x - \lambda^2\right] - \frac{\lambda^2}{2}} dx$$

$$= e^{-\lambda^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+i\lambda)^2} dx \overset{z = x + i\lambda}{=}$$



$$= e^{-\lambda^2/2} \lim_{n \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{C_n} e^{-z^2/2} dz$$

Como $e^{-z^2/2}$ é holomorfa, $\int_\gamma e^{-z^2/2}\,dz = 0$,



onde $\gamma = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\int_{C_1} e^{-z^2/2}\,dz + \underbrace{\int_{C_2} e^{-z^2/2}\,dz}_{\to\, 0} + \underbrace{\int_{C_3} e^{-z^2/2}\,dz}_{\to\, -\sqrt{2\pi}} + \underbrace{\int_{C_4} e^{-z^2/2}\,dz}_{\to\, 0} = 0$$

b) $$\hat{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\lambda} f(ax)\,dx \overset{ax=y}{=}$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-i\frac{y}{a}\lambda} f(y)\,\frac{dy}{a} = \frac{1}{a}\hat{f}\left(\frac{\lambda}{a}\right)$$

c) $$\mathcal{F}[e^{-tx^2}] = \mathcal{F}\left[e^{-(\sqrt{2t}\cdot x)^2/2}\right]$$

$$\overset{\substack{\text{usando} \\ a:b}}{=} \frac{1}{\sqrt{2t}} e^{-\left(\frac{\lambda}{\sqrt{2t}}\right)^2 \frac{1}{2}} = \frac{1}{\sqrt{2t}} e^{-\frac{\lambda^2}{4t}}$$

3 Passando transformadas,

$$\widehat{\partial_t u} = \partial_t \hat{u} = \widehat{\partial_x^2 u} = -\lambda^2 \hat{u}$$

EDO

$$\Rightarrow \hat{u}(t,\lambda) = e^{-\lambda^2 t}\, \hat{u}_0(\lambda)$$

$$\Rightarrow u(t,\lambda) = \widecheck{e^{-\lambda^2 t}\, \hat{u}_0(\lambda)}$$

$$\Rightarrow u(t,\lambda) = \widecheck{e^{-\lambda^2 t}} * u_0(x) \cdot \frac{1}{\sqrt{2\pi}}$$

Q2

$$\Rightarrow u(t,\lambda) = \frac{e^{-x^2/4t}}{\sqrt{2t}} * u_0(x) \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} * u_0(x)$$

4 $\hat{\mu}(f) = \mu(\hat{f}) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(x)\,dx\, \mu(dt)$

Fubini

$$= \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \frac{e^{-ixt}}{\sqrt{2\pi}}\, \mu(dt)\, dx = \int_{\mathbb{R}} f(x)\, \frac{\phi_\mu(-x)}{\sqrt{2\pi}}\, dx$$

$\Rightarrow \hat{\mu} = T_g$, onde $g(t) = \frac{\phi_\mu(-t)}{\sqrt{2\pi}}$. Como a transformada é única, μ é única.