Teoria Espectral Prova 1 2015.1

Gabarito

(1) a) $t>0$, $\rho(tx) = \inf\{\lambda;\ tx \in \lambda C\} =$

$= \inf\{\lambda;\ x \in \frac{\lambda}{t} C\} = \inf\{\lambda' t;\ x \in \frac{\lambda' t}{t} C\}$

$= \inf\{\lambda' t;\ x \in \lambda' C\} = t\,\rho(x)$

b) Fixe $\varepsilon > 0$ e sejam $r = \rho(x) + \varepsilon$ e $s = \rho(y) + \varepsilon$.

$\Rightarrow \frac{x}{r} \in C$ e $\frac{y}{s} \in C$. Daí, como C é convexo,

$$\left(\frac{r}{r+s}\right) \cdot \frac{x}{r} + \left(\frac{s}{r+s}\right) \cdot \frac{y}{s} \in C$$

$$\Rightarrow \frac{1}{r+s} \cdot (x+y) \in C \Rightarrow \rho(x+y) \leq r + s.$$

Como ε é arbitrário, $\rho(x+y) \leq \rho(x) + \rho(y)$.

(2) a) $f' \equiv 0$ b) $f' = 2\delta_0$ c) $f'(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

(3) $(fT)'(g) = -(fT)(g') = -T(fg') = -T((fg)' - f'g)$

$= -T((fg)') + T(f'g) = T'(fg) + (f'T)(g)$

$= (fT')(g) + (f'T)(g)$.

④ Seja $f \in L^p(\mathbb{R})$, $1 < p < +\infty$. $|T_f(g)| = \left|\int_{\mathbb{R}} f(x) g(x)\, dx\right|$

$$\leq \int_{\mathbb{R}} |f(x) g(x)|\, dx \overset{\text{Hölder}}{\leq} \|f\|_p \cdot \|g\|_q, \quad \text{onde} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Temos que $\|g\|_q^q = \int_{\mathbb{R}} |g(x)|^q\, dx = \int_{\mathbb{R}} |g(x)|^q \cdot \frac{(1+|x|^k)^q}{(1+|x|^k)^q}\, dx$

$$\leq \left(\sup_x |g(x)| \cdot (1+|x|^k)\right)^q \cdot \int_{\mathbb{R}} \frac{dx}{(1+|x|^k)^q}$$

$$\leq \left(\|g\|_{0,0} + \|g\|_{0,k}\right)^q \cdot C$$ e substitui.

⑤ $\left|\int \varphi_n(x) f(x)\, dx - f(0)\right| = \left|\int \varphi_n(x) (f(x) - f(0))\, dx\right|$

$$\leq \int |\varphi_n(x) (f(x) - f(0))|\, dx = \int_{|x|>a} |\varphi_n(x) (f(x) - f(0))|\, dx$$

$$+ \int_{|x|\leq a} |\varphi_n(x) (f(x) - f(0))|\, dx \leq 2\|f\|_\infty \int_{|x|>a} |\varphi_n(x)|\, dx \; +$$

$$+ \max_{|x|\leq a} |f(x) - f(0)| \cdot \int_{\mathbb{R}} \varphi_n(x)\, dx \longrightarrow 0$$

⑥ Argumente que $f(x) = e^{-x^2} \in \mathcal{S}(\mathbb{R})$. Entretanto

$$T_{e^{x^2}}\left(e^{-x^2}\right) = \int_{\mathbb{R}} e^{x^2} \cdot e^{-x^2}\, dx = +\infty, \text{ ou seja,}$$

nem está definido.

⑦ $$(T_f)'(g) = -T_f(g') = -\int_{\mathbb{R}} f(x)\, g'(x)\, dx =$$

$$= -\int_0^1 \frac{1}{\sqrt{x}}\, g'(x)\, dx$$

$$dv = g'(x)\,dx$$
$$v = g(x) - g(0)$$

$$u = \frac{1}{\sqrt{x}}$$

Daí, $$-\int_0^1 \frac{1}{\sqrt{x}}\, g'(x)\, dx = \lim_{\varepsilon \downarrow 0} -\int_\varepsilon^1 \frac{1}{\sqrt{x}}\, g'(x)\, dx =$$

$$= \lim_{\varepsilon \downarrow 0} -\frac{1}{\sqrt{x}}\left(g(x) - g(0)\right)\Big|_\varepsilon^1 + \int_\varepsilon^1 \frac{-1}{2x^{3/2}}\left(g(x) - g(0)\right) dx$$

$$= g(0) - g(1) + \lim_{\varepsilon \downarrow 0} \frac{g(\varepsilon) - g(0)}{\sqrt{\varepsilon}}$$

$$+ \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \frac{g(0) - g(x)}{2x^{3/2}}\, dx$$

$$= \delta_0(g) - \delta_1(g) + \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \frac{g(0) - g(x)}{2x^{3/2}}\, dx$$