Probabilidade 2014.2 Prova 3 18/12/14

GABARITO RESUMIDO

① a) $X = X_1 + \cdots + X_n$, X_i iid $\sim$ Bernoulli(p)

$$\varphi_{X_i}(t) = \mathbb{E}[e^{itX_i}] = pe^{it} + (1-p)$$

$$\Rightarrow \varphi_X(t) = [pe^{it} + (1-p)]^n$$

b) $$\varphi_X'(t) = n[pe^{it} + (1-p)]^{n-1} ipe^{it}$$

$$\varphi_X''(t) = n(n-1)[pe^{it} + (1-p)]^{n-2}(ipe^{it})^2$$
$$+ n[pe^{it} + (1-p)]^{n-1} i^2 pe^{it}$$

$$\varphi_X''(0) = n(n-1)(-p^2) + (-np)$$

$$= -n^2p^2 + np^2 - np$$

$$\Rightarrow \mathbb{E}X^2 = n^2p^2 - np^2 + np = np(np - p + 1)$$

② a) $$\varphi_Y(t) = \frac{1}{2}\int_{-1}^{1} e^{itx}\,dx = \frac{\operatorname{sen} t}{t}$$

b) $$\varphi_{Y_n}(t) = \prod_{k=1}^{n} \varphi_{X_k/2^k}(t) = \prod_{k=1}^{n} \mathbb{E}\left[e^{\frac{itX_k}{2^k}}\right]$$

$$= \prod_{k=1}^{n} \cos\left(\frac{t}{2^k}\right) = \prod_{k=1}^{n} \frac{\operatorname{sen}(t/2^{k-1})}{2\operatorname{sen}(t/2^k)} = \operatorname{sen} t \cdot \frac{1}{2^n \operatorname{sen}(t/2^n)}$$

$$\longrightarrow \frac{\operatorname{sen} t}{t}$$

③ a) $X \sim N(0,1)$. $X_n = \begin{cases} X, & n \text{ par} \\ -X, & n \text{ ímpar} \end{cases}$

b) $\mathbb{P}[X_n \leq a] \longrightarrow \begin{cases} 0, & a < c \\ 1, & a > c \end{cases}$

Logo, $\mathbb{P}[X_n < a] \longrightarrow 0, \quad a < c$

e $\mathbb{P}[X_n > a] \longrightarrow 0, \quad a > c$

Daí, $\mathbb{P}[|X_n - c| > \varepsilon] = \mathbb{P}[X_n > c + \varepsilon] +$

$\mathbb{P}[X < c - \varepsilon] \longrightarrow 0$.

④ $X \sim \text{Poisson}(\lambda)$. Logo, $\varphi_X(t) = e^{\lambda(e^{it}-1)}$

$$\varphi_{X_n}(t) = [p_n e^{it} + (1-p_n)]^n = \left[1 + \frac{n p_n (e^{it}-1)}{n}\right]^n$$

$$\xrightarrow[n \to +\infty]{} e^{\lambda(e^{it}-1)}$$